WELL-POSEDNESS FOR A SYSTEM OF TRANSPORT AND DIFFUSION EQUATIONS IN MEASURE SPACES WITH NON-LOCAL FLOWS AND SOURCE TERMS

AZMY S. ACKLEH AND NICOLAS SAINTIER

Abstract. In this paper, we establish the well-posedness of the following system of two transport equations coupled with a diffusion equation:

$$\begin{aligned} &\partial_t \mu_t^1 + \nabla \cdot (v^1[\mu_t]\mu_t^1) = N^1(t,\mu_t), \\ &\partial_t \mu_t^2 - \Delta \mu_t^2 = N^2(t,\mu_t), \\ &\partial_t \mu_t^3 + \nabla \cdot (v^2[\mu_t]\mu_t^3) = N^3(t,\mu_t), \end{aligned}$$

in $\mathbb{R}^d$ where $\mu_t^1, \mu_t^2, \mu_t^3$ are finite signed measures. Here, the vector field v^1, v^2 and the source term N^1, N^2, N^3 depend on the measure-valued solution vector $\mu_t = (\mu_t^1, \mu_t^2, \mu_t^3)$.

Key words: System of transport and diffusion equations, Measure-valued solutions, Well-posedness, Fixed point, Bounded-Lipschitz norm

2010 Mathematics Subject Classifications: 35M30, 35F20, 35K57, 35A01, 37C25

1. Introduction

Transport equations appear naturally in physics as they express in mathematical terms a conservation law. Considering measure-valued solution allows, in particular, to study in a unified framework discrete as well as continuous dynamics. This explains why they are ubiquitous in modeling of systems of interacting agents/particles in many applications including physics, social sciences, and biology (e.g. [3, 8, 9, 10, 14, 48, 49]).

To further motivate the study of a system of transport and diffusion equations on the space of measures that we undertake in this paper, in what follows we provide some examples from various area focusing on the modeling aspects. We do not pretend to list all the possible applications but only to emphasize that measure-valued functions provide a natural framework to model many phenomena.

When considering n interacting particles at the microscopic level in the mean-field setting, in the sense that each particle is subject to the same mean force from the other particles, we can write the evolution of the state $x_i(t)$ of the i-th particle as

$$\frac{d}{dt}x_i(t) = \frac{1}{n}\sum_{j=1}^{n} K(x_i(t), x_j(t)) \tag{1}$$

where K models the interaction between particles. It is then easy to see that the empirical measure $\mu_t^n := \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(t)}$ solves, in a weak sense, the transport equation

$$\partial_t \mu + \nabla \cdot (v[\mu_t]\mu_t) = 0 \tag{2}$$

where the vector-field $v[\mu_t]$ is given by $v[\mu_t](x) = \int_{\mathbb{R}^d} K(x,y)\, d\mu_t(y)$ and represents the cumulated influence of the whole distribution μ_t at the point x. As the number n of particles goes to infinity, we expect this equation to be the right limit of the microscopic ODE system. In fact, well-posedness can be proved by standard techniques - we refer the reader to [6, 9, 19, 25, 36, 37, 48].

More recently this kind of equations proved to be useful in the modeling of some social processes including formation of opinion. A large group of individuals, each one of them having an opinion on some given issue, is interacting. The individuals involved in an interaction modify their opinion following some given mechanism. The goal is then to analyze the time evolution of the distribution of opinions in the population. One way of achieving this consists of writing down a Boltzmann-like equation satisfied by the distribution of opinions. Assuming that collisions are binary and result in only a very small change of opinions for each participant, it can be proved that the long-time behavior of the Boltzmann-like equation can be well approximated by a transport equation of the form (2). This approach is well-known in statistical physics and has been adapted to the context of opinion formation process by Toscani [49]. We refer to [3, 39, 40, 41, 44, 45] for examples of theoretical studies of some particular equations like (2) appearing in this context. We refer to the book [38] for more on the application and theory of kinetic models in social and economic sciences.

In population biology, the authors in [8, 10, 14, 15, 47] considered equations similar to (2), in the context of measure-valued solutions, with or without a source term. The authors in [14, 15, 47] focus their attention on the dynamics of cells aggregation. They recall that displacement of an individual is not the result of only the application of forces following the law of mechanics but also depend on the interaction with the external environment, e.g., other cells or chemical fields or extracellular matrix components. Furthermore, in extremely viscous regimes such as those arising in some biological environments as e.g. cells in highly viscous fluids or in an high cellular density environment (e.g., phytoplankton cell population in the ocean or a virus population spreading in the air), Newton equations of motion can be written in the overdamped approximation so that the velocity of moving individuals and not their acceleration is typically proportional to the sensed forces. In that case the movement of the i-th cell located at $x_i(t) \in \mathbb{R}^d$ at time t can be modeled by an equation similar to (1) but with an additional term $T(c_t)$ accounting for the influence of the external environment:

$$\frac{d}{dt}x_i(t) = T[c_t](x) + \frac{1}{n}\sum_{j=1}^{n} K(x_i(t), x_j(t)). \tag{3}$$

Thinking of c_t as the concentration of some chemicals at time t, we can model its time evolution by a diffusion equation like

$$\partial_t c_t(x) = \Delta c_t(x) - dc_t(x) + S(t, x) \tag{4}$$

where $d > 0$ is the degradation (or death/evaporation) rate and S is a source term. Notice that if the chemicals are secreted by the cells themselves then S should depend on the distribution μ_t of the cells at time t. A possible choice is $S_t = b\mu_t$ for some birth/production rate $b > 0$. In such a setting we are naturally led to consider a coupled system of the form

$$\begin{aligned} &\partial_t \mu_t + \nabla \cdot (v\mu_t) = 0 \\ &\partial_t c_t(x) = \Delta c_t(x) - dc_t(x) + b\mu_t \end{aligned} \tag{5}$$

with the vector-field

$$v(t, x) = v[\mu_t, c_t](x) = T(c_t)(x) + \int_{\mathbb{R}^d} K(x, y)\, d\mu_t(y). \tag{6}$$

Notice that μ_t and c_t are both measures. This system was considered in [47]. In that paper the authors developed a numerical scheme for solving this system but the problem of its well-posedness was left open. The authors in [47] emphasize that the framework of measure-valued function allows naturally a multiscale analysis intertwining the cellular (microscopic) and the multicellular (macroscopic) levels. Indeed we can write μ_t as the sum of a continuous part modeling the cellular aggregate as a continuum and a finite sum of Dirac masses representing particular cells.

A system with two coupled transport equations for measure-valued solution is also relevant from the applied point of view. Indeed, the authors in [15] argued that "in a wide range of pattern formations, characteristic of biological processes, large aggregates of non-specialized inactivated cells are collectively guided by a small number of specialized and activated individuals [...]. This happens for instance in angiogenesis, morphogenesis, and wound healing mechanisms, or in the metastatic infiltration of solid tumors". These consideration led them to model the distribution μ^u of the aggregate of undifferentiated cells by a continuous measure and the distribution μ^d

of the few differentiated cells by a discrete distribution. The time evolution of both measures μ^u and μ^d is then adequately modeled by a system of conservation laws of the form

$$\begin{aligned} \partial_t\mu_t^d + \nabla\cdot(v^d\mu_t^d) = 0 \\ \partial_t\mu_t^u + \nabla\cdot(v^u\mu_t^u) = 0 \end{aligned} \tag{7}$$

where the vector-fields v^d and v^u drive the displacement of the specialized and undifferentiated cells. Notice that the displacement of a differentiated cell results from interaction with other differentiated cells as well as interaction with undifferentiated cells. The same applies to an undifferentiated cell. It is thus natural to assume that v^d and v^u depends on μ^d and μ^u i.e. $v^d = v^d[\mu_t^d, \mu_t^u]$ and $v^u = v^u[\mu_t^d, \mu_t^u]$.

Notice that a system like (7) also appears in opinion formation process, see, e.g., [18, 41]. In these papers the authors model opinion formation in heterogeneous populations starting from a system of Boltzmann-like equations which becomes, after a suitable rescaling of the parameters, a system similar to (7). The long-time behavior is then investigated either numerically or theoretically.

The goal of this paper is to give a unified treatment for the well-posedness of systems (5) and (7) described above. We will do so by considering the following general transport-diffusion system:

$$\begin{aligned} &\partial_t\mu_t^1 + \nabla\cdot(v^1[\mu_t]\mu_t^1) = N^1(t,\mu_t), \\ &\partial_t\mu_t^2 - \Delta\mu_t^2 = N^2(t,\mu_t) \\ &\partial_t\mu_t^3 + \nabla\cdot(v^2[\mu_t]\mu_t^3) = N^3(t,\mu_t), \end{aligned} \tag{8}$$

in $\mathbb{R}^d$ where $\mu_t = (\mu_t^1, \mu_t^2, \mu_t^3)$ is a vector of measures. Here, the vector-fields v^1, v^2 and the source terms N^1, N^2, N^3 depend on μ. The well-posedness of system (8) will be the focus of this paper. Here we apply a fixed point argument to establish the main result. Because of the nonlinearities in the vector fields and the source terms this requires the development of techniques to establish novel estimates on auxiliary linear transport and diffusion problems. These estimates are then utilized to show that the nonlinear fixed point operator is contractive.

The paper is organized as follows: in Section 2 we present some preliminary material. In Section 3 we present a statement of the main result and provide a comparison between our result and those already available in the literature. Sections 4 and 5 are devoted to establishing auxiliary results on the well-posedness of a linear diffusion equation and a linear transport equation in measure spaces, respectively. These results are used in Section 6 to prove the main theorem of the paper and establish the well-posedness of the new system (8) proposed here which combines transport and diffusion equations in measure spaces. In Section 7 we present some corollaries to the main theorem along with some remarks. Finally, we provide a sketch of the proof of Proposition 4.1 in the appendix.

2. Preliminaries

We denote by $M_b(\mathbb{R}^d)$ the space of finite signed Borel measures. We will work with two different norms on $M_b(\mathbb{R}^d)$: the Total Variation (TV) norm and the Bounded Lipschitz (BL) norm. The TV norm $\|\mu\|_{TV}$ of a measure $\mu \in M_b(\mathbb{R}^d)$ is

$$\|\mu\|_{TV} = \sup_{\phi \in C_c(\mathbb{R}^d),\, \|\phi\|_\infty \leq 1} \int_{\mathbb{R}^d} \phi \, d\mu = |\mu|(\mathbb{R}^d). \tag{9}$$

Here, $|\mu| = \mu^+ + \mu^-$, with μ^+ and μ^- being the positive and negative parts of μ as given by the Jordan decomposition.

To define the BL norm we first need to introduce the space $W^{1,\infty}(\mathbb{R}^d)$ consisting of bounded Lipschitz functions ϕ, namely $\phi \in W^{1,\infty}(\mathbb{R}^d)$ if ϕ is bounded and there exists $C > 0$ such that

$$|\phi(x) - \phi(y)| \leq C|x - y| \qquad \text{for any } x, y \in \mathbb{R}^d.$$

We denote by $Lip(\phi)$ the least admissible constant C. We endow $W^{1,\infty}(\mathbb{R}^d)$ with the norm $\|\phi\|_{W^{1,\infty}} := \max\{\|\phi\|_\infty, Lip(\phi)\}$. The bounded Lipschitz norm (or Dudley norm, or Fortet-Mourier norm) $\|\mu\|_{BL}$ of a measure $\mu \in M_b(\mathbb{R}^d)$ is then defined by

$$\|\mu\|_{BL} = \sup \Big\{ \int_{\mathbb{R}^d} \phi \, d\mu : \, \phi \in W^{1,\infty}(\mathbb{R}^d), \, \|\phi\|_{W^{1,\infty}} \leq 1 \Big\}.$$

This norm is well-studied in probability theory and has also been used to study well-posedness of transport equations in population biology (e.g., [8]).

Recall that a sequence $(\mu_n)_n \subset M_b(\mathbb{R}^d)$ converges weakly to $\mu \in M_b(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \phi \, d\mu_n \to \int_{\mathbb{R}^d} \phi \, d\mu$ for any $\phi \in C_b(\mathbb{R}^d)$, where $C_b(\mathbb{R}^d)$ denotes the space of bounded continuous functions on $\mathbb{R}^d$. It is known that in that case the sequence $(\mu_n)_n$ has bounded TV norm and is also tight in the sense that for any $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that for any n, $|\mu_n|(\mathbb{R}^d \backslash K) \leq \varepsilon$. (see [20][Thm 4]).

Here it is worth pointing out that the weak convergence and the BL convergence are not equivalent in general. It is known that if $\mu_n \to \mu$ weakly then $\|\mu_n - \mu\|_{BL} \to 0$ (see [20][Thm 6]) but the converse is, in general, false if we are not working with non-negative measures, in which case both convergences are equivalent (see [20][Thm 8]). Indeed, there are sequences converging to 0 in the BL norm but that are neither tight nor TV-bounded. Consider for instance $\mu_n = \sqrt{n}(\delta_{n+1/n} - \delta_n) \in M_b(\mathbb{R})$. Then $\|\mu_n\|_{BL} = \frac{\sqrt{n}}{n} \to 0$ but $(\mu_n)_n$ is neither tight nor TV-bounded (since $\|\mu_n\|_{TV} = 2\sqrt{n}$).

In this paper, unless otherwise specified, we will always endow $M_b(\mathbb{R}^d)$ with the BL-norm. We recall that the space $M_b(\mathbb{R}^d)$ is complete when endowed with the TV norm but not with the BL norm. However a subspace of the form

$$M_{b,R}(\mathbb{R}^d) := \{\mu \in M_b(\mathbb{R}^d) : \|\mu\|_{TV} = |\mu|(\mathbb{R}^d) \leq R\}, \tag{10}$$

where $R > 0$, is complete when endowed with the BL norm (see e.g. Thm 2.7 in [27]).

We recall that the push-forward of $\mu \in M_b(\mathbb{R}^d)$ through a Borel measurable map $f : \mathbb{R}^d \to \mathbb{R}^d$ is the measure $f\#\mu$ defined on Borel sets $B \subset \mathbb{R}^d$ by $f\#\mu(B) = \mu(f^{-1}(B))$. It can be easily verified that $(f\#\mu, \phi) = (\mu, \phi \circ f)$, where $(\mu, \phi) = \int_{\mathbb{R}^d} \phi \, d\mu$ denotes the natural pairing between measures $\mu \in M_b(\mathbb{R}^d)$ and bounded measurable functions ϕ.

3. The Main Result

We consider the well-posedness of system (8) with an initial condition $\mu_0 = (\mu_0^1, \mu_0^2, \mu_0^3) \in M_b(\mathbb{R}^d)^3$. This system is a particular example of systems of transport and diffusion equations in the space of measures that can be considered. Indeed, we could have also considered systems where μ^2 satisfies a transport equation or where μ^3 satisfies a diffusion equation. It will be clear from the proof that our result of well-posedness for the particular system (8) holds with similar assumptions in the case of the other systems of coupled transport and diffusion equations in the space of measures mentioned above.

Let us now state our assumptions on the vector-fields v^i, $i = 1, 2$, and the source terms N^k, $k = 1, 2, 3$. We assume that the vector-fields

$$v^1, v^2 : M_b(\mathbb{R}^d)^3 \to W^{1,\infty}(\mathbb{R}^d)$$

are continuous from $M_b(\mathbb{R}^d)^3$ to $L^\infty(\mathbb{R}^d)$ and for any $R > 0$ there exist constants $L_R^v, C_R^v > 0$ such that for any $\mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d)^3$ and any $i = 1, 2$, the following is satisfied:

(V1) $\|v^i[\mu] - v^i[\tilde{\mu}]\|_\infty \leq L_R^v \|\mu - \tilde{\mu}\|_{BL}$,
(V2) $\|v^i[\mu]\|_{W^{1,\infty}} \leq C_R^v$.

We assume that the source terms

$$N^k : \mathbb{R}_+ \times M_b(\mathbb{R}^d)^3 \to M_b(\mathbb{R}^d) \qquad k = 1, 2, 3,$$

are continuous in (t,μ) (recall that we endow $M_b(\mathbb{R}^d)$ with the BL norm) and that for any $R>0$, there exist constants $L_R^N, C_R^N>0$ such that for any $t\in\mathbb{R}$, any $\mu,\tilde{\mu}\in M_{b,R}(\mathbb{R}^d)^3$ and any $k=1,2,3$, the following is satisfied:

(N1) $\|N^k(t,\mu)-N^k(t,\tilde{\mu})\|_{BL}\le L_R^N\|\mu-\tilde{\mu}\|_{BL}$,
(N2) $\|N^k(t,\mu)\|_{TV}\le C_R^N$.

For $T>0$ denote,

$$M_T=\{\mu\in C([0,T],M_b(\mathbb{R}^d)^3) \text{ with } \mu^2\in L^1((0,T)\times\mathbb{R}^d)\}.$$

We state the definition of weak solution of system (8):

Definition 3.1. *We say $\mu\in M_T$ is a solution of system (8) on $[0,T]$ with initial condition $\mu_0\in M_b(\mathbb{R}^d)^3$ if $\sup_{0\le t\le T}\|\mu_t\|_{TV}<\infty$ and for every $\phi=(\phi^1,\phi^2,\phi^3)\in C_c^1([0,T]\times\mathbb{R}^d)\times C_c^2([0,T]\times\mathbb{R}^d)\times C_c^1([0,T]\times\mathbb{R}^d)$, the following equations are satisfied for any $t\in[0,T]$:*

(11)
$$\begin{aligned}
&\int_{\mathbb{R}^d}\phi^1(t,x)d\mu_t^1(x)-\int_{\mathbb{R}^d}\phi^1(0,x)d\mu_0^1(x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}(\partial_t\phi^1(s,x)+v^1[\mu_s](x)\nabla\phi^1(s,x))d\mu_s^1(x)ds+\int_0^t\int_{\mathbb{R}^d}\phi^1(s,x)dN^1(s,\mu_s)(x)ds,\\
&\int_{\mathbb{R}^d}\phi^2(t,x)d\mu_t^2(x)-\int_{\mathbb{R}^d}\phi^2(0,x)d\mu_0^2(x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}(\partial_t\phi^2(s,x)+\Delta\phi_2(s,x))d\mu_s^2(x)ds+\int_0^t\int_{\mathbb{R}^d}\phi^2(s,x)dN^2(s,\mu_s)(x)ds,\\
&\int_{\mathbb{R}^d}\phi^3(t,x)d\mu_t^3(x)-\int_{\mathbb{R}^d}\phi_3(0,x)d\mu_0^3(x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}(\partial_t\phi^3(s,x)+v^2[\mu_s](x)\nabla\phi^3(s,x))d\mu_s^3(x)ds+\int_0^t\int_{\mathbb{R}^d}\phi^3(s,x)dN^3(s,\mu_s)(x)ds.
\end{aligned}$$

We now state the following result on the well-posedness of system (8):

Theorem 3.1. *For any initial condition $\mu_0\in M_b(\mathbb{R}^d)^3$ there exists T^* and a unique solution $\mu\in M_T$, $T<T^*$, with $T^*<\infty$ iff $\lim_{t\to T^*,t<T^*}\|\mu_t\|_{TV}=\infty$. Furthermore, this solution is continuous with respect to the initial condition in the following sense: Let $\mu,\tilde{\mu}$ be two solutions defined on $[0,T]$ corresponding to initial conditions μ_0 and $\tilde{\mu}_0$, respectively, and assume that*

$$\|\mu_t\|_{TV},\|\tilde{\mu}_t\|_{TV}\le R \qquad \text{for } t\in[0,T].$$

Then

$$\|\mu_t-\tilde{\mu}_t\|_{BL}\le r(t)\|\mu_0-\tilde{\mu}_0\|_{BL}, \tag{12}$$

where $r(t)$ is a continuous non-decreasing function depending only on $L^v_R, C^v_R, L^N_R, C^N_R$ and satisfying $r(0) = 1$.

We end this section with a comparison between our result and previous results concerning the well-posedness in the space of measures of a single transport equation or a system of transport equations. Concerning the case of a single transport equation of the form

$$\partial_t \mu_t + \nabla \cdot (v[\mu_t]\mu_t) = N(t, \mu_t), \tag{13}$$

we refer to the papers [8, 10, 22, 23, 27, 28, 35, 42]. The authors in [8] established the well-posedness of (13) with a general source term satisfying the same assumptions as ours and a vector-field $v = v(x)$ independent of μ_t. The need for such a source term is illustrated by various examples in population dynamics including the well studied selection-mutation models (e.g., [1, 13]). On the other hand the authors in [42] studied (13) with a general vector-field satisfying assumptions similar to ours in spirit (though assumption (V1) is stated with a generalized Wasserstein distance $W^{a,b}_p$ introduced by the authors which is equivalent to the BL distance when $p = 1$ (see [43])). The source term $N = N(\mu)$ they consider is required to satisfy assumptions similar to (N1)-(N2). They also require $N(\mu)$, $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, to be absolutely continuous with respect to the Lebesgue measure and be supported in a given ball $B_0(R)$, for a fixed $R > 0$ independent of μ. This assumption, which is stronger than ours, is motivated by applications in the modeling of pedestrian flows and useful for the constructive proof they develop which is based on an Euler scheme similar to the standard Euler scheme applied for ODEs. In [22, 23] the authors study the existence-uniqueness of mild solutions to a transport equation with a linear source term. In [22] they consider a given vector field independent of the solution similar to [8] while in [23] they extend the results to a vector field that depends on the solution. The author in [35] consider, as a particular case of a whole theory of ordinary differential equations in metric spaces, equations like (13) with a general vector-field $v[t, \mu]$ and a special form source term $N(t, \mu) = \bar{N}(t, \mu)\mu$ which is important from the point of view of biological applications (we also consider this form of source term in (39) below). Under assumptions on v and $\bar{N}$ similar to ours, existence and uniqueness of solutions are proved.

In [27] and [28] the authors study a size-structured population model of the form

$$\begin{aligned} &\partial_t \mu_t + \partial_x (F_2(t, \mu_t)\mu_t) = F_3(t, \mu_t)\mu_t \qquad \text{in } \mathbb{R}_+ \times [0, T], \\ &F_2(t, \mu_t)(0)\mu_t(0) = \int_0^{+\infty} F_1(t, \mu_t)(x)\, d\mu_t(x) \qquad \text{in } (0, T]. \end{aligned} \tag{14}$$

In our framework, this corresponds to taking the velocity field $v[t,\mu] = F_2(t,\mu_t)$ and the source term $N(t,\mu) = F_3(t,\mu)\mu + F[t,\mu]\delta_{x=0}$ with $F[t,\mu] := \int_0^{+\infty} F_1(t,\mu)(x)\, d\mu(x)$. The authors in [27] and [28] establish the well-posedness of (14) using two different proofs under assumptions compatible with ours. Continuity of the solution with respect to F_1, F_2 and F_3 is also proved, an issue we do not consider in this paper.

Concerning a system of transport equation, we mention [16], [24] and [50]. The authors in [16] establish the well-posedness of a system of transport equations without source terms and with measure-dependent vector-fields of the form $v[\mu](t,x) = V(t,x,\eta * \mu)$ where $V(t,x,r)$ is globally bounded and Lipschitz in (x,r) uniformly in t, and $\eta(t,x)$ is globally bounded and Lipschitz in x uniformly in t. So, except for the time dependence, this v satisfies assumptions (V1)-(V2). The result in [24] falls under those in [16] but the method of proof is different and based on gradient flows in Wasserstein space. Finally, the author in [50] considers a two-sex model described by a system of coupled age-structured equations (of similar form to (14) but with $F_2 \equiv 1$) and establishes its well-posedness under assumptions implying ours.

4. Well-posedness for linear diffusion equation in measure spaces

We now give a preliminary result on the solution of the linear heat equation with measure data. In particular, we consider the equation

$$\begin{aligned} &\partial_t u - \Delta u = \sigma \qquad \text{in } (0,+\infty)\times\mathbb{R}^d \\ &u_{|t=0} = u_0, \end{aligned} \tag{15}$$

where $u_0 \in M_b(\mathbb{R}^d)$ and $\sigma \in M_b(Q)$ where $Q = (0,+\infty)\times\mathbb{R}^d$. We let $Q_T = (0,T)\times\mathbb{R}^d$, $T > 0$.

We understand this equation in the following sense: a function u is a solution if $u \in L^1(Q_T)$ for any $T > 0$ and if for any $\phi \in C_c^{1,2}([0,+\infty)\times\mathbb{R}^d)$ the following holds:

$$\iint_Q (\partial_t + \Delta)\phi(t,x)\, u(t,x) dt dx + \int_{\mathbb{R}^d} \phi(0,x)\, du_0(x) + \iint_Q \phi\, d\sigma = 0. \tag{16}$$

Here, $\phi \in C_c^{1,2}([0,+\infty)\times\mathbb{R}^d)$ means that (i) $\phi(\cdot,x)$ is C^1 (continuously differentiable) for any $x \in \mathbb{R}^d$, (ii) $\phi(t,\cdot)$ is C^2 (twice-continuously differentiable) for any $t \geq 0$ and (iii) ϕ has a compact support.

We denote by $K(t,x) = (4\pi t)^{-d/2}\exp(-|x|^2/(4t))$ the heat kernel and by P_t the associated semi-group given by $P_t\mu(x) = (K(t,\cdot) * \mu)(x)$.

The following result is most probably known but we could not find an explicit reference. So for the reader's convenience, we include in the Appendix a short sketch of the proof.

Proposition 4.1. *For any $u_0 \in M_b(\mathbb{R}^d)$ and any $\sigma \in M_b(Q)$ there exists a unique solution to* (15) *which is given explicitly by*

$$u(t,x) = (P_t u_0)(x) + (K * \sigma)(t,x). \tag{17}$$

Moreover, the map $t' \to u(t',x)dx$ is continuous from $[0,+\infty)$ to $M_b(\mathbb{R}^d)$ in the weak convergence (against test-functions in $C_b(\mathbb{R}^d)$), and thus also in the BL-norm, at any t such that $|\sigma|(\{t\}\times\mathbb{R}^d) = 0$.

Remark 4.1. *We prove in fact that the solution u given by* (17) *verifies a slightly stronger statement than* (16) *namely that for any $t > 0$, and any $\phi \in C_c^{1,2}([0,+\infty)\times \mathbb{R}^d)$,*

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(t,x)u(t,x)\,dx - \int_{\mathbb{R}^d} \phi(0,x)u_0(x)\,dx \\ &= \int_0^t \int_{\mathbb{R}^d} (\partial_t + \Delta)\phi(t,x)\; u(t,x)dtdx + \int_0^t \int_{\mathbb{R}^d} \phi\, d\sigma. \end{aligned} \tag{18}$$

Here, we will mainly be interested in the case where σ has the following special form

$$\sigma = \int_0^T \delta_s \otimes \tilde{\sigma}_s\, ds \tag{19}$$

for some $T > 0$ and $\tilde{\sigma} : [0,T] \to M_b(\mathbb{R}^d)$. Here $\delta_s \otimes \tilde{\sigma}_s$ denotes the product measure on Q defined for any $\phi \in C_b(Q)$ by $(\delta_s \otimes \tilde{\sigma}_s, \phi) = \int \phi(s,x)\, d\tilde{\sigma}_s(x)$, and (19) means that

$$\int_Q \phi\, d\sigma := \int_0^T \int_{\mathbb{R}^d} \phi(s,x)\, d\tilde{\sigma}_s(x)\, ds \qquad \text{for any } \phi \in C_b(Q). \tag{20}$$

We now have the following corollary for this special case.

Corollary 4.1. *Let $\tilde{\sigma} : [0,T] \to M_b(\mathbb{R}^d)$ be continuous in the BL norm and bounded in the TV norm, i.e., $\|\tilde{\sigma}_s\|_{TV} \le R$ for any $s \in [0,T]$. Then the integral in* (19) *is a Bochner integral in $\overline{M_b}([0,T]\times\mathbb{R}^d)$, the completion of $M_b([0,T]\times\mathbb{R}^d)$ under the BL norm. Moreover, $\|\sigma\|_{TV} \le RT$,*

$$(\sigma,\phi) = \int_0^T (\hat{\sigma}_s, \phi(s,\cdot))\, ds \qquad \text{for any } \phi \in L^\infty([0,T]\times\mathbb{R}^d), \tag{21}$$

and

$$|\sigma|(\{t\}\times\mathbb{R}^d) = 0 \qquad \text{for any } t \in [0,T]. \tag{22}$$

Furthermore, for any initial condition $u_0 \in M_b(\mathbb{R}^d)$ there exists a unique solution to (15) in $L^1((0,T)\times\mathbb{R}^d)$ which is given explicitly by

$$u(t,x) = (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}^d} K(t-s, x-y)\, d\tilde{\sigma}_s(x) ds,$$

and the map $t \to u(t,x)dx$ is continuous from $[0,T]$ to $M_b(\mathbb{R}^d)$ in the weak convergence and thus also in the BL-norm.

Remark 4.2. *It follows from Remark 4.1 that for any $t > 0$, and any $\phi \in C_c^2([0,+\infty)\times\mathbb{R}^d)$,*

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(t,x)u(t,x)\,dx &- \int_{\mathbb{R}^d} \phi(0,x)u_0(x)\,dx \\ &= \int_0^t \int_{\mathbb{R}^d} (\partial_t + \Delta)\phi(t,x)\; u(t,x)dtdx + \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\, d\tilde{\sigma}(s,x)ds.\end{aligned} \tag{23}$$

Proof of Corollary 4.1. Consider the measure $\hat{\sigma}_s$ on $[0,T]\times\mathbb{R}^d$ given by $\hat{\sigma}_s = \delta_s \otimes \tilde{\sigma}_s$, i.e., $\int_{[0,T]\times\mathbb{R}^d} \phi\, d\hat{\sigma}_s = \int_{\mathbb{R}^d} \phi(s,x)\, d\tilde{\sigma}_s(x)$ for $\phi \in C_b([0,T]\times\mathbb{R}^d)$. Notice that $\|\hat{\sigma}_s\|_{BL} = \|\tilde{\sigma}_s\|_{BL}$, $\|\hat{\sigma}_s\|_{TV} = \|\tilde{\sigma}_s\|_{TV} \le R$, and $\hat{\sigma}_s$ is continuous in s in the BL norm. Indeed, for any $\phi \in W^{1,\infty}([0,T]\times\mathbb{R}^d)$, we have

$$\begin{aligned}&|\int_{[0,T]\times\mathbb{R}^d} \phi(s,x)\, d\hat{\sigma}_s(x) - \int_{[0,T]\times\mathbb{R}^d} \phi(s,x)\, d\hat{\sigma}_t(x)| \\ &\quad \le \int_{[0,T]\times\mathbb{R}^d} |\phi(s,x) - \phi(t,x)|\, d|\tilde{\sigma}_s|(x) + |\int_{[0,T]\times\mathbb{R}^d} \phi(t,x)\, d(\tilde{\sigma}_s - \tilde{\sigma}_t)(x)| \\ &\quad \le Lip(\phi)(|s-t|R + \|\tilde{\sigma}_s - \tilde{\sigma}_t\|_{BL}).\end{aligned}$$

Hence, we obtain

$$\|\hat{\sigma}_s - \hat{\sigma}_t\|_{BL} \le Lip(\phi)(|s-t|R + \|\tilde{\sigma}_s - \tilde{\sigma}_t\|_{BL}$$

which goes to 0 as $s \to t$.

Denote by $\overline{M_b}([0,T]\times\mathbb{R}^d)$ the completion of $M_b([0,T]\times\mathbb{R}^d)$ under the BL norm. We will verify that the integral $\sigma := \int_0^T \hat{\sigma}_s\, ds$ is well-defined as a Bochner integral of $\overline{M_b}([0,T]\times\mathbb{R}^d)$-valued functions.

According to [17][Thm 2], we have to prove that (i) $\int_0^T \|\hat{\sigma}_s\|_{BL}\, ds < \infty$, and that (ii) the map $s \to \hat{\sigma}_s$ is strongly-measurable (i.e., it is the limit a.e. of simple functions). Point (i) is easy because the map $s \to \|\hat{\sigma}\|_{BL}$ is continuous (and thus measurable) and bounded by $\|\hat{\sigma}\|_{TV} \le R$. Concerning point (ii) we use [22][Appendix C1] which states that (ii) is equivalent to proving that for every $\phi \in L^\infty([0,T]\times\mathbb{R}^d)$ the map $F(s) := \int \phi\, d\hat{\sigma}_s$ is measurable. Let $\phi_\varepsilon := \phi * \rho_\varepsilon$ where ρ_ε are the standard mollifiers. Then $\phi_\varepsilon \to \phi$ in $L^\infty_{loc}([0,T]\times\mathbb{R}^d)$ and ϕ_ε is bounded Lipschitz for any ε with $\|\phi_\varepsilon\|_\infty \le \|\phi\|_\infty$. It follows that $F_\varepsilon(s) := \int \phi_\varepsilon\, d\hat{\sigma}_s$ is continuous in s (for a fixed ε). Moreover, since for any s, the measure $\hat{\sigma}_s$ is tight, we have that $F_\varepsilon(s) \to F(s)$ as $\varepsilon \to 0$. Thus, F is measurable as a pointwise limit of continuous functions. Another possibility to prove (ii) would be to notice that $\overline{M_b}([0,T] \times \mathbb{R}^d)$ is separable so that the strong-measurability is equivalent to the weak measurability (see [17][Thm 2]), i.e., the map $s \to (\psi, \tilde{\sigma}_s)$ is measurable for any $\psi \in \overline{M_b}([0,T] \times \mathbb{R}^d)'$, the dual of $\overline{M_b}([0,T] \times \mathbb{R}^d)$. Then use [31][Thm 7] which states that the dual of $\overline{M_b}([0,T] \times \mathbb{R}^d)$ is isometrically isomorphic to $W^{1,\infty}([0,T] \times \mathbb{R}^d)$ by $S(\psi)(x) := \psi(\delta_x)$.

According to [17][Cor.8], $\int_0^T \hat{\sigma}_s \frac{ds}{T}$ belongs to the closure (in the BL norm) of the convex hull of $\{\hat{\sigma}_s\}_{0 \le s \le T}$. Since $\hat{\sigma}_s \in M_{b,R}([0,T] \times \mathbb{R}^d)$ for any $0 \le s \le T$ and $M_{b,R}([0,T] \times \mathbb{R}^d)$ is convex and closed under the BL norm, we deduce that $\int_0^T \hat{\sigma}_s \frac{ds}{T} \in M_{b,R}([0,T] \times \mathbb{R}^d)$ and then that $\int_0^T \hat{\sigma}_s \, ds \in M_{b,RT}([0,T] \times \mathbb{R}^d)$.

Given $\phi \in W^{1,\infty}([0,T]\times\mathbb{R}^d)$ the expression $F(\mu) := \int_{[0,T]\times\mathbb{R}^d} \phi \, d\mu$ defines a bounded linear form on $\overline{M_b}([0,T]\times\mathbb{R}^d)$. Thus by [17][Thm 6] we have $F(\int_0^T \hat{\sigma}_s \, ds) = \int_0^T (F, \hat{\sigma}_s) \, ds$, i.e.,

$$(\sigma, \phi) = \int_0^T (\hat{\sigma}_s, \phi) \, ds \qquad \phi \text{ bounded Lipschitz.}$$

On the other hand consider $\nu : \phi \in C_b([0,T] \times \mathbb{R}^d) \to \int_0^T (\hat{\sigma}_s, \phi) \, ds$. This is well-defined because we already saw that the integrand is measurable in s and is bounded in absolute value by $\|\phi\|_\infty R$. Thus, ν is a bounded linear form on $C_b([0,T] \times \mathbb{R}^d)$, i.e., a bounded measure. In particular, $\nu(E) = \int_0^T \hat{\sigma}_s(E) \, ds$ for any $E \subset \mathbb{R}^d$ Borel. Then the measures ν and σ coincide on the bounded Lipschitz functions. It follows from [20][Lemma 6] that they are equal as measures. Thus,

$$(\sigma, \phi) = \int_0^T (\hat{\sigma}_s, \phi) \, ds \qquad \phi \in L^\infty(\mathbb{R}^d).$$

In particular, let $\phi = 1_{(t-\delta, t+\delta)}$, then

$$|\sigma|((t-\delta, t+\delta) \times \mathbb{R}^d) \le \int_{t-\delta}^{t+\delta} |\tilde{\sigma}_s|(\mathbb{R}^d) \, ds \le 2R\delta.$$

Letting $\delta \to 0$ we obtain (22). The last part of this corollary is a direct consequence of Proposition 4.1. □

5. Well-posedness of linear transport equation in measure space

Given $T > 0$, we consider the equation

$$(24) \qquad \begin{aligned} \partial_t \mu_t + \nabla \cdot (b(t,x)\mu_t) &= \sigma_t \qquad 0 < t \le T, \\ \mu_{|t=0} &= \mu_0, \end{aligned}$$

where $b(t,\cdot)$ is a vector-field and $\sigma_t \in M_b(\mathbb{R}^d)$. When $b = b(x)$ this equation has been studied in [8] when the right-hand side can depend on μ_t. Here, we adapt their framework to the case of time-dependent vector-fields. This is an improvement that will be needed to study the well-posedness of the general system (8).

We say that μ is a solution on $[0,T]$ if $\mu : [0,T] \to M_b(\mathbb{R}^d)$ is continuous in the BL norm and bounded in the TV norm, i.e., $\sup_{0\le t\le T} \|\mu_t\|_{TV} < \infty$, $\mu_{|t=0} = \mu_0$, and for any $0 < t \le T$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \phi \, d\mu_t = \int_{\mathbb{R}^d} b(t,x)\nabla\phi(x)\, d\mu_t + \int_{\mathbb{R}^d} \phi \, d\sigma_t \qquad \phi \in C_c^1(\mathbb{R}^d). \tag{25}$$

Notice that we can equivalently replace (25) by either

$$\int_{\mathbb{R}^d} \phi\, d\mu_t - \int_{\mathbb{R}^d} \phi \, d\mu_0 = \int_0^t \int_{\mathbb{R}^d} b(s,x)\nabla\phi(x)\, d\mu_s \, ds + \int_0^t \int_{\mathbb{R}^d} \phi\, d\sigma_s \, ds \qquad \phi \in C_c^1(\mathbb{R}^d), \tag{26}$$

or, for any $\phi \in C_c^1([0,T]\times\mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(t,x)\, d\mu_t(x) - \int_{\mathbb{R}^d} \phi(0,x)\, d\mu_0(x) \\ &= \int_0^t \int_{\mathbb{R}^d} \partial_t\phi(s,x) + b(s,x)\nabla\phi(s,x)\, d\mu_s\, ds + \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\, d\sigma_s(x)\, ds, \end{aligned} \tag{27}$$

this last formulation being the one used in Definition 3.1.

We denote by $\mathcal{T}_{s,t}$ the flow associated with $b(t,x)$, namely, for any $x \in \mathbb{R}^d$, $\mathcal{T}_{s,t}(x)$ satisfies

$$\begin{aligned} \frac{d}{dt}\mathcal{T}_{s,t}(x) &= b(t, \mathcal{T}_{s,t}(x)), \\ \mathcal{T}_{s,s}(x) &= x, \end{aligned}$$

and for simplicity of notation we let $\mathcal{T}_t := \mathcal{T}_{0,t}$.

To assure the global existence of $\mathcal{T}_{s,t}$, we assume for simplicity that b is continuous in (t,x) and globally Lipschitz in x uniformly in t, i.e., there exists a constant $L^b > 0$ such that

$$|b(t,x) - b(t,x')| \le L^b |x - x'| \qquad \text{for any } t \in [0,T] \text{ and any } x, x' \in \mathbb{R}^d. \tag{28}$$

Then $\mathcal{T}_{s,t}$ is well-defined in $\mathbb{R}^d$ for any $T \ge t \ge s \ge 0$.

Proposition 5.1. *Assume that b is continuous, bounded (i.e. there exists a constant $C^b > 0$ such that $|b(t,x)| \le C^b$ for any $t \in [0,T]$ and $x \in \mathbb{R}^d$), and that b satisfies (28). Consider $\sigma \in C([0,T], M(\mathbb{R}^d))$ such that $\sup_{0\le t\le T} \|\sigma_t\|_{TV} < \infty$. Then, for any initial condition $\mu_0 \in M_b(\mathbb{R}^d)$, equation (24) has a unique solution which is defined* *on* $[0,T]$ *and is given explicitly by*

$$\mu_t = \mathcal{T}_t\sharp\mu_0 + \int_0^t \mathcal{T}_{s,t}\sharp\sigma_s\, ds. \tag{29}$$

Proof. Uniqueness is easy since it is known that for any $t>0$ the zero function is the only solution to (24) with $\mu_0=0$ and $\sigma_t=0$ (see e.g. [51]). We thus have to prove that μ given by (29) is well-defined and is a solution.

Notice that for any $t>0$, the expression $\int_0^t \mathcal{T}_{s,t}\sharp\sigma_s\, ds$ defines a bounded measure in $\mathbb{R}^d$. Indeed for any $\phi\in C(\mathbb{R}^d)$, $\|\phi\|_\infty\le 1$, we have

$$|(\mathcal{T}_{s,t}\sharp\sigma_s,\phi)| = |(\sigma_s,\phi\circ\mathcal{T}_{s,t})| \le \|\sigma_s\|_{TV}\|\phi\circ\mathcal{T}_{s,t}\|_\infty \le R_\sigma$$

where $R_\sigma := \sup_{t\ge 0}\|\sigma_t\|_{TV}$, so that $\|\mathcal{T}_{s,t}\sharp\sigma_s\|_{TV}\le R_\sigma$. Moreover, the measures $\mathcal{T}_{s,t}\sharp\sigma_s$ are continuous in s for the BL norm. Indeed, for $\phi\in W^{1,\infty}(\mathbb{R}^d)$ with $\|\phi\|_{W^{1,\infty}}\le 1$, we have

$$\begin{aligned}\Big|(\mathcal{T}_{s,t}\sharp\sigma_s,\phi)-(\mathcal{T}_{s',t}\sharp\sigma_{s'},\phi)\Big| &= \Big|\int_{\mathbb{R}^d}\phi(\mathcal{T}_{s,t}(x))\,d\sigma_s(x)-\int_{\mathbb{R}^d}\phi(\mathcal{T}_{s',t}(x))\,d\sigma_{s'}(x)\Big|\\ &\le \int_{\mathbb{R}^d}|\mathcal{T}_{s,t}(x)-\mathcal{T}_{s',t}(x)|\,d\sigma_s(x)+\|\phi\circ\mathcal{T}_{s',t}\|_{W^{1,\infty}}\|\sigma_s-\sigma_{s'}\|_{BL}.\end{aligned}$$

Independently it is easily seen that $Lip(\mathcal{T}_{s,t})\le e^{L^b|t-s|}$ and then

$$\begin{aligned}|\mathcal{T}_{s,t}(x)-\mathcal{T}_{s',t}(x)| &= |\mathcal{T}_{s,t}(x)-\mathcal{T}_{s,t}(\mathcal{T}_{s',s}(x))|\le e^{L^b|t-s|}|x-\mathcal{T}_{s',s}(x)|\\ &\le L^b e^{L^b|t-s|}|s-s'|.\end{aligned}$$

Taking the supremum over all such ϕ, we obtain

$$\Big\|\mathcal{T}_{s,t}\sharp\sigma_s-\mathcal{T}_{s',t}\sharp\sigma_{s'}\Big\|_{BL}\le L^b e^{L^b|t-s|}|s-s'|R_\sigma+e^{L^b|t-s'|}\|\sigma_s-\sigma_{s'}\|_{BL},$$

which goes to 0 as $s'\to s$.

It is now easily seen by adapting the proof of Corollary 4.1 that $\int_0^t \mathcal{T}_{s,t}\sharp\sigma_s\, ds$ is a Bochner integral defining a bounded measure on $\mathbb{R}^d$ with total variation less than tR_σ and satisfies

$$\Big(\int_0^t \mathcal{T}_{s,t}\sharp\sigma_s\, ds,\phi\Big)=\int_0^t(\mathcal{T}_{s,t}\sharp\sigma_s,\phi)\,ds=\int_0^t(\sigma_s,\phi\circ\mathcal{T}_{s,t})\,ds$$

for any $\phi\in L^\infty(\mathbb{R}^d)$.

We verify that μ defined by (29) is continuous in t. For t',t and $\phi\in W^{1,\infty}(\mathbb{R}^d)$ with $\|\phi\|_{W^{1,\infty}}\le 1$, we have

$$\begin{aligned}(\mu_{t'}-\mu_t,\phi) &= \int_{\mathbb{R}^d}\phi(\mathcal{T}_{t'}(x))-\phi(\mathcal{T}_t(x))\,d\mu_0(x)+\int_t^{t'}(\sigma_s,\phi\circ\mathcal{T}_{s,t'})\,ds\\ &\quad+\int_0^t(\sigma_s,\phi\circ\mathcal{T}_{s,t'}-\phi\circ\mathcal{T}_{s,t})\,ds.\end{aligned}$$

Since ϕ is 1-Lipschitz and $|(\sigma_s, \phi \circ \mathcal{T}_{s,t'})| \leq \|\sigma_s\|_{TV} \|\phi \circ \mathcal{T}_{s,t'}\|_\infty \leq R_\sigma$, we have

$$\begin{aligned} |(\mu_{t'} - \mu_t, \phi)| &\leq \int_{\mathbb{R}^d} |\mathcal{T}_{t'}(x)) - \mathcal{T}_t(x)|\, d|\mu_0|(x) + |t'-t| R_\sigma \\ &\quad + \int_0^t \int_{\mathbb{R}^d} |\mathcal{T}_{s,t'}(x) - \mathcal{T}_{s,t}(x)|\, d|\sigma_s|(x)\, ds. \end{aligned}$$

Taking the supremum over all ϕ using that

$$|\mathcal{T}_{s,t'}(x) - \mathcal{T}_{s,t}(x)| \leq \int_t^{t'} |b(\tau, \mathcal{T}_{s,\tau}(x))|\, d\tau \leq L^b(t'-t) \tag{30}$$

we obtain

$$\|\mu_{t'} - \mu_t\|_{BL} \leq |t'-t| (L^b |\mu_0|(\mathbb{R}^d) + R_\sigma + t R_\sigma L^b)$$

which goes to 0 as $t' \to t$.

Next, we verify that (25) holds. Given $\phi \in C_c^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \phi\, d\mu_t = \int_{\mathbb{R}^d} \phi(\mathcal{T}_t(x))\, d\mu_0 + \int_0^t \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s,t}(x))\, d\sigma_s(x)\, ds.$$

Notice that $\int_{\mathbb{R}^d} \phi(\mathcal{T}_{s,t}(x))\, d\sigma_s(x)$ is continuous in s. To see this we first write

$$\begin{aligned} &| \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s,t}(x))\, d\sigma_s(x) - \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s',t}(x))\, d\sigma_{s'}(x)| \\ &= | \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s,t}(x)) - \phi(\mathcal{T}_{s',t}(x))\, d\sigma_s(x) + \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s',t}(x))\, d(\sigma_s - \sigma_{s'})(x)| \\ &\leq \int_{\mathbb{R}^d} |\mathcal{T}_{s,t}(x)) - \mathcal{T}_{s',t}(x)|\, d|\sigma_s|(x) + \|\phi \circ \mathcal{T}_{s',t}\|_{W^{1,\infty}} \|\sigma_{s'} - \sigma_s\|_{BL}. \end{aligned}$$

Then,

$$\begin{aligned} &| \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s,t}(x))\, d\sigma_s(x) - \int_{\mathbb{R}^d} \phi(\mathcal{T}_{s',t}(x))\, d\sigma_{s'}(x)| \\ &\leq R_\sigma L^b e^{L^b|t-s|} |s-s'| + e^{L^b|t-s|} \|\sigma_{s'} - \sigma_s\|_{BL}, \end{aligned}$$

from which we deduce the continuity in s recalling that σ_s is continuous in s. Hence,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \phi\, d\mu_t &= \int_{\mathbb{R}^d} \nabla\phi(x) b(t,x)\, d(\mathcal{T}_t \sharp \mu_0) + \int_{\mathbb{R}^d} \phi(\mathcal{T}_{t,t}(x))\, d\sigma_t(x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \nabla\phi(x) b(t,x)\, d(\mathcal{T}_{s,t} \sharp \sigma_s)(s)\, ds \\ &= \int_{\mathbb{R}^d} \phi(x)\, d\sigma_t(x) + \int_{\mathbb{R}^d} \nabla\phi(x) b(t,x)\, d\mu_t(x) \end{aligned}$$

which is (25). □

Remark 5.1. *We assumed that b is globally bounded and globally Lipschitz to obtain existence of a global solution and also because we have a priori no control on the size of the support of μ_0 and σ_s. If we assume that μ_0 and σ_s are all supported in some compact set in $\mathbb{R}^d$, then we may only assume that b is locally bounded and* *locally Lipschitz but in that case we will only have a priori existence of a local in time solution.*

6. Well-posedness for the full system: proof of Theorem 3.1

In this section we prove the well-posedness of the system (8) given by

$$\begin{aligned}&\partial_t\mu_t^1+\nabla\cdot(v^1[\mu_t]\mu_t^1)=N^1(t,\mu_t),\\&\partial_t\mu_t^2=\Delta\mu_t^2+N^2(t,\mu_t),\\&\partial_t\mu_t^3+\nabla\cdot(v^2[\mu_t]\mu_t^3)=N^3(t,\mu_t),\end{aligned}\tag{31}$$

where $\mu_t=(\mu_t^1,\mu_t^2,\mu_t^3)\in(M_b(\mathbb{R}^d))^3$ and $\mu_t^2\in L^1(\mathbb{R}^d)$. We let $\|\mu_t\|_{BL}=\sum_{k=1}^3\|\mu_t^k\|_{BL}$ and $\|\mu_t\|_{TV}=\sum_{k=1}^3\|\mu_t^k\|_{TV}$.

Notice that if $\mu\in M_T$ is a solution then the maps $t\in[0,T]\to N^k(t,\mu_t)\in M_b(\mathbb{R}^d)$, $k=1,2,3$, are continuous and, since $\sup_{0\le t\le T}\|\mu_t^k\|_{TV}<\infty$, the vector-fields $b_k(t,x):=v^k[\mu_t](x)$, $k=1,2$, are globally Lipschitz in x uniformly in t by $(V2)$. Then according to Corollary 4.1 (here we use that $\mu^2\in L^1((0,T)\times\mathbb{R}^d)$) and Proposition 5.1, μ must be a fixed-point of the map $\Gamma(\mu)=(\Gamma^1(\mu),\Gamma^2(\mu),\Gamma^3(\mu))$ defined by

$$\begin{aligned}\Gamma^1(\mu)_t&=\mathcal{T}_t^{v^1[\mu]}\sharp\mu_0^1+\int_0^t\mathcal{T}_{s,t}^{v^1[\mu]}\sharp N^1(s,\mu_s)\,ds,\\\Gamma^2(\mu)_t&=P_t\mu_0^2+\int_0^tP_{t-s}N^2(s,\mu_s)\,ds,\\\Gamma^3(\mu)_t&=\mathcal{T}_t^{v^2[\mu]}\sharp\mu_0^3+\int_0^t\mathcal{T}_{s,t}^{v^2[\mu]}\sharp N^3(s,\mu_s)\,ds.\end{aligned}$$

Conversely, if $\mu\in C([0,T],M_b(\mathbb{R}^d)^3)$ is a fixed-point of Γ such that $\sup_{0\le t\le T}\|\mu_t\|_{TV}<\infty$ then μ is a solution. Then the vector-fields $b^k(t,x):=v^k[\mu_t](x)$ is bounded by (V2) and is also continuous in (t,x) and globally Lipschitz in x uniformly in $t\in[0,T]$ since

$$\begin{aligned}|b^k(t,x)-b^k(t',x')|&\le|v^k[\mu_t](x)-v^k[\mu_{t'}](x)|+|v^k[\mu_{t'}](x)-v^k[\mu_{t'}](x')|\\&\le L_R^v\|\mu_t-\mu_{t'}\|_{BL}+C_R^v|x-x'|.\end{aligned}$$

Moreover $\tilde\sigma_t^k:=N^k(t,\mu_t)$ is a sequence of measures bounded in TV-norm by (N2) and is also continuous in t for the weak convergence since N^k is continuous in (t,μ). We can then rewrite the equality $\mu_t=\Gamma(\mu)_t$ as

$$\begin{aligned}\mu_t^1&=\mathcal{T}_t^{b^1}\sharp\mu_0^1+\int_0^t\mathcal{T}_{s,t}^{b^1}\sharp\tilde\sigma_s^1\,ds,\\\mu_t^2&=P_t\mu_0^2+\int_0^tP_{t-s}\tilde\sigma_s^2\,ds,\\\mu_t^3&=\mathcal{T}_t^{b^2}\sharp\mu_0^3+\int_0^t\mathcal{T}_{s,t}^{b^2}\sharp\tilde\sigma_s^3\,ds.\end{aligned}$$

It then follows from Corollary 4.1 and Proposition 5.1 that μ is a solution in the sense of Definition 3.1. Notice that $\Gamma^2(\mu)$ belongs to $L^1((0,T)\times\mathbb{R}^d)$ so that $\mu^2 \in L^1((0,T)\times\mathbb{R}^d)$.

We thus have to prove that Γ has a unique fixed point in the space

$$X = \{\mu \in C([0,T], M_b(\mathbb{R}^d)^3) : \mu_{|t=0} = \mu_0, \|\mu_t\|_{TV} \le 2\|\mu_0\|_{TV} \,\forall t \in [0,T]\} \tag{32}$$

for a given positive T. Here X is endowed with the sup-norm $\|\mu\|_X := \max_{0\le t\le T}\|\mu_t\|_{BL}$ and hence is complete.

We first prove some properties of Γ.

Lemma 6.1. *If $\mu \in C([0,T], M_b(\mathbb{R}^d)^3)$ is such that $\sup_{0\le t\le T}\|\mu_t\|_{TV} < \infty$, then $\Gamma(\mu) \in C([0,T], M_b(\mathbb{R}^d)^3)$.*

Proof. We need to show that each component $\Gamma^k(\mu)$ is continuous in t. We begin by proving this for the case $k=1$. To this end, let $b(t,x) := v[\mu_t](x)$ and $\sigma_t := N^1(t,\mu_t)$. Then σ is continuous in t and $\sup_{0\le t\le T}\|\sigma_t\|_{TV} < \infty$ since N^1 satisfies (N2). Moreover, b is bounded and globally Lipschitz in x uniformly in t since v^1 satisfies (V2), and continuous in (t,x) since

$$\begin{aligned} |b(t,x) - b(t',x')| &\le |v^1[\mu_t](x) - v^1[\mu_{t'}](x)| + |v^1[\mu_{t'}](x) - v^1[\mu_{t'}](x')| \\ &\le L_R^v\|\mu_t - \mu_{t'}\|_{BL} + C_R^v|x - x'| \end{aligned}$$

where $R := \sup_{0\le t\le T}\|\mu_t\|_{TV}$. Therefore, the continuity of $\Gamma^1(\mu)$ then follows from the proof of Proposition 5.1. The continuity of $\Gamma^3(\mu)$ is proved in a similar manner.

Concerning the continuity of $\Gamma^2(\mu)$, we let $\sigma_t = N^2(t,\mu_t)$. Then as before σ is continuous in t and $\sup_{0\le t\le T}\|\sigma_t\|_{TV} < \infty$. The result then follows from Corollary 4.1. □

Lemma 6.2. *Let $\mu, \tilde\mu \in C([0,T], M_b(\mathbb{R}^d)^3)$ and $R > 0$ such that $\|\mu_t\|_{TV}, \|\tilde\mu_t\|_{TV} \le R$ for any $0 \le t \le T$. Then*

$$\|\mathcal{T}_{s,t}^{v[\mu]} - \mathcal{T}_{s,t'}^{v[\mu]}\|_\infty \le C_R^v|t'-t|, \qquad Lip(\mathcal{T}_{s,t}^{v[\mu]}) \le e^{C_R^v|t-s|}, \tag{33}$$

and

$$\|\mathcal{T}_{s,t}^{v[\mu]} - \mathcal{T}_{s,t}^{v[\tilde\mu]}\|_\infty \le L_R^v e^{C_R^v|t-s|}\int_s^t \|\mu_\tau - \tilde\mu_\tau\|_{BL}\,d\tau. \tag{34}$$

Proof. The proofs of (33) and (34) are standard and similar. Let us prove (34). We denote $v(t,x)=v[\mu_t](x)$, $\tilde{v}(t,x)=v[\tilde{\mu}_t](x)$, $\mathcal{T}_{s,t}:=\mathcal{T}_{s,t}^{v[\mu]}$ and $\tilde{\mathcal{T}}_{s,t}:=\mathcal{T}_{s,t}^{v[\tilde{\mu}]}$. First

$$\mathcal{T}_{s,t}(x)-\tilde{\mathcal{T}}_{s,t}(x)=\int_s^t v(\tau,\mathcal{T}_{s,\tau}(x))-\tilde{v}(\tau,\tilde{\mathcal{T}}_{s,\tau}(x))\,d\tau$$

$$=\int_s^t v(\tau,\mathcal{T}_{s,\tau}(x))-\tilde{v}(\tau,\mathcal{T}_{s,\tau}(x))\,d\tau+\int_s^t \tilde{v}(\tau,\mathcal{T}_{s,\tau}(x))-\tilde{v}(\tau,\tilde{\mathcal{T}}_{s,\tau}(x))\,d\tau.$$

Then by (V1) and (V2),

$$|\mathcal{T}_{s,t}(x)-\tilde{\mathcal{T}}_{s,t}(x)| \;\leq\; L_R^v\int_s^t \|\mu_\tau-\tilde{\mu}_\tau\|_{BL}+C_R^v\int_s^t |\mathcal{T}_{s,\tau}(x)-\tilde{\mathcal{T}}_{s,\tau}(x)|\,d\tau.$$

The result now follows from Gronwall's lemma. □

Lemma 6.3. *Given $R>0$, consider $\mu,\tilde{\mu}\in C([0,T],M_{b,R}(\mathbb{R}^d)^3)$ with initial conditions μ_0 and $\tilde{\mu}_0$. Then*

$$\|\Gamma(\mu)_t-\Gamma(\tilde{\mu})_t\|_{BL}\leq A_R\int_0^t \|\mu_s-\tilde{\mu}_s\|_{BL}\,ds+e^{C_R^v t}\|\mu_0-\tilde{\mu}_0\|_{BL}, \tag{35}$$

where $A_R=2e^{C_R^v t}(L_R^v\|\mu_0\|_{TV}+\frac{C_R^N L_R^v}{C_R^v}+L_R^N)+L_R^N$.

Proof. We denote $\mathcal{T}_{s,t}^1:=\mathcal{T}_{s,t}^{v^1[\mu]}$, $\tilde{\mathcal{T}}_{s,t}^1:=\mathcal{T}_{s,t}^{v^1[\tilde{\mu}]}$. Given $\phi\in W^{1,\infty}(\mathbb{R}^d)$, $\|\phi\|_{W^{1,\infty}}\leq 1$, we have

$$\begin{aligned}&|(\Gamma^1(\mu)_t-\Gamma^1(\tilde{\mu})_t,\phi)|\\ &\leq |(\mu_0^1,\phi\circ\mathcal{T}_t^1-\phi\circ\tilde{\mathcal{T}}_t^1)|+|(\mu_0^1-\tilde{\mu}_0^1,\phi\circ\tilde{\mathcal{T}}_t^1)|+\int_0^t |(N^1(s,\mu_s),\phi\circ\mathcal{T}_{s,t}^1-\phi\circ\tilde{\mathcal{T}}_{s,t})|\,ds\\ &\quad+\int_0^t |(N^1(s,\mu_s)-N^1(s,\tilde{\mu}_s),\phi\circ\tilde{\mathcal{T}}_{s,t}^1)|\,ds\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

We estimate I_k, $k=1,..,4$, as follows. First in view of (34)

$$|I_1|\leq\int_{\mathbb{R}^d}|\mathcal{T}_t^1(x)-\tilde{\mathcal{T}}_t^1(x)|\,d|\mu_0^1|(x)\leq L_R^v e^{C_R^v t}\|\mu_0^1\|_{TV}\int_0^t \|\mu_\tau-\tilde{\mu}_\tau\|_{BL}\,d\tau,$$

$$\begin{aligned}|I_3| &\leq \int_0^t\int_{\mathbb{R}^d}|\mathcal{T}_{s,t}^1(x)-\tilde{\mathcal{T}}_{s,t}^1(x)|\,d|N^1(s,\mu_s)|(x)\,ds\\ &\leq \int_0^t \|N^1(s,\mu_s)\|_{TV}L_R^v e^{C_R^v(t-s)}\int_s^t \|\mu_\tau-\tilde{\mu}_\tau\|_{BL}\,d\tau\,ds\\ &\leq \frac{C_R^N L_R^v}{C_R^v}(e^{C_R^v t}-1)\int_0^t \|\mu_\tau-\tilde{\mu}_\tau\|_{BL}\,d\tau.\end{aligned}$$

Independently (33) gives $Lip(\phi\circ\tilde{\mathcal{T}}_{s,t}^1)\leq Lip(\phi)Lip(\tilde{\mathcal{T}}_{s,t}^1)\leq e^{C_R^v|t-s|}$ so that $\|\phi\circ\tilde{\mathcal{T}}_{s,t}^1\|_{W^{1,\infty}}\leq\max\{\|\phi\|_\infty,Lip(\phi\circ\tilde{\mathcal{T}}_{s,t}^1))\leq e^{C_R^v|t-s|}$. It follows that

$$I_2\leq\|\mu_0^1-\tilde{\mu}_0^1\|_{BL}\|\phi\circ\tilde{\mathcal{T}}_t^1\|_{W^{1,\infty}}\leq e^{C_R^v t}\|\mu_0^1-\tilde{\mu}_0^1\|_{BL}.$$

and also using (N1),

$$I_4 \leq L_R^N \int_0^t e^{C_R^v(t-s)} \|\mu_s - \tilde{\mu}_s\|_{BL}\, ds \leq L_R^N e^{C_R^v t} \int_0^t \|\mu_s - \tilde{\mu}_s\|_{BL}\, ds.$$

It follows that

$$\begin{aligned} \|\Gamma^1(\mu)_t - \Gamma^1(\tilde{\mu})_t\|_{BL} &\leq e^{C_R^v t}(L_R^v \|\mu_0^1\|_{TV} + \frac{C_R^N L_R^v}{C_R^v} + L_R^N) \int_0^t \|\mu_s - \tilde{\mu}_s\|_{BL}\, ds \\ &\quad + e^{C_R^v t} \|\mu_0^1 - \tilde{\mu}_0^1\|_{BL}. \end{aligned}$$

The same estimate holds for $\|\Gamma^3(\mu)_t - \Gamma^3(\tilde{\mu})_t\|_{BL}$ replacing $\mu_0^1, \tilde{\mu}_0^1$ by $\mu_0^3, \tilde{\mu}_0^3$.

Concerning Γ^2 we write

$$\begin{aligned} (\Gamma^2(\mu)_t - \Gamma^2(\tilde{\mu})_t, \phi) &= (P_t(\mu_0^2 - \tilde{\mu}_0^2), \phi) + \int_0^t (P_{t-s}(N^2(s, \mu_s) - N^2(s, \tilde{\mu}_s)), \phi)\, ds \\ &= (\mu_0^2 - \tilde{\mu}_0^2, P_t \phi) + \int_0^t (N^2(s, \mu_s) - N^2(s, \tilde{\mu}_s)), P_{t-s}\phi)\, ds \end{aligned}$$

and then

$$\begin{aligned} |(\Gamma^2(\mu)_t - \Gamma^2(\tilde{\mu})_t, \phi)| &\leq \|\mu_0^2 - \tilde{\mu}_0^2\|_{BL} \|P_t \phi\|_{W^{1,\infty}} \\ &\quad + \int_0^t \|N^2(s, \mu_s) - N^2(s, \tilde{\mu}_s)\|_{BL} \|P_{t-s}\phi\|_{W^{1,\infty}}\, ds \end{aligned}$$

which can be bounded in a similar way as I_2 and I_4 above but using standard properties of $_t$. We obtain

$$\|\Gamma^2(\mu)_t - \Gamma^2(\tilde{\mu})_t\|_{BL} \leq \|\mu_0^2 - \tilde{\mu}_0^2\|_{BL} + L_R^N \int_0^t \|\mu_s - \tilde{\mu}_s\|_{BL}\, ds.$$

Hence, we are able deduce (35). □

Lemma 6.4. *For T small enough (depending on $\|\mu_0\|_{TV}$ only), $\Gamma(X) \subset X$ and Γ is a strict contraction in X.*

Proof. Let $\mu \in X$. In view of Lemma 6.3, we known that $\Gamma(\mu)$ is continuous. Let us show that $\|\Gamma(\mu)_t\|_{TV} \leq 2\|\mu_0\|_{TV}$ for $t \in [0, T]$. Let $R = 2\|\mu_0\|_{TV}$, so that $\|\mu_t\|_{TV} \leq R$, and $\mathcal{T}_{s,t}^1 := \mathcal{T}_{s,t}^{v^1[\mu]}$, $\mathcal{T}_{s,t}^2 := \mathcal{T}_{s,t}^{v^2[\mu]}$. Notice that $\|N(t, \mu_t)\|_{TV} \leq C_R^N$ for $t \in [0, T]$. For $\phi \in C(\mathbb{R}^d)$, $\|\phi\|_\infty \leq 1$, we have

$$\begin{aligned} |(\Gamma^1(\mu)_t, \phi)| &\leq |(\mu_0^1, \phi \circ \mathcal{T}_t^1)| + \int_0^t |(N^1(s, \mu_s), \phi \circ \mathcal{T}_{s,t}^1)\, ds \\ &\leq \|\mu_0^1\|_{TV} \|\phi \circ \mathcal{T}_t^1\|_\infty + \int_0^t \|N^1(s, \mu_s)\|_{TV} \|\phi \circ \mathcal{T}_{s,t}^1\|_\infty\, ds \end{aligned}$$

so that

$$\|\Gamma^1(\mu)_t\|_{TV} \leq \|\mu_0^1\|_{TV} + C_R^N T \qquad \text{for } t \in [0, T].$$

The same estimate holds true for $\|\Gamma^3(\mu)_t\|_{TV}$. Concerning $\Gamma^2(\mu)$ we have

$$\begin{aligned}|(\Gamma^2(\mu)_t,\phi)| &\le |(\mu_0^2,P_t\phi)|+\int_0^t |(N^2(s,\mu_s),P_{t-s}\phi)\,ds \\ &\le \|\mu_0^2\|_{TV}\|P_t\phi\|_\infty+\int_0^t \|N^2(s,\mu_s)\|_{TV}\|P_{t-s}\phi\|_\infty\,ds.\end{aligned}$$

Using that $\|P_t\phi\|_\infty\le\|\phi\|_\infty\le 1$, we obtain

$$\|\Gamma^2(\mu)_t\|_{TV}\le\|\mu_0^2\|_{TV}+C_R^N T \qquad \text{for } t\in[0,T].$$

Thus, for any $t\in[0,T]$,

$$\|\Gamma(\mu)_t\|_{TV}\le\|\mu_0\|_{TV}+3C_R^N T,$$

which is less than $2\|\mu_0\|_{TV}$ choosing T such that $3C_R^N T\le\|\mu_0\|_{TV}$. In particular T depend only on $\|\mu_0\|_{TV}$.

Let us now show that Γ is a strict contraction for T small. Consider $\mu,\tilde\mu\in X$. Since $\mu_0=\tilde\mu_0$, Lemma 6.3 with $R=2\|\mu_0\|_{TV}$ gives

$$\|\Gamma(\mu)_t-\Gamma(\tilde\mu)_t\|_{BL}\le\{2e^{C_R^v t}(L_R^v\|\mu_0\|_{TV}+\frac{C_R^N L_R^v}{C_R^v}+L_R^N)+L_R^N\}\int_0^t\|\mu_s-\tilde\mu_s\|_{BL}\,ds$$

for any $t\in[0,T]$. Then, it follows that

$$\|\Gamma(\mu)-\Gamma(\tilde\mu)\|_X\le\{2e^{C_R^v T}(L_R^v\|\mu_0\|_{TV}+\frac{C_R^N L_R^v}{C_R^v}+L_R^N)+L_R^N\}T\|\mu-\tilde\mu\|_X.$$

Thus, we can choose T small enough depending only on R (and thus on $\|\mu_0\|_{TV}$) so that Γ is a strict contraction in X. □

It follows that Γ has a unique fixed-point in X. We deduce the existence and uniqueness of a solution defined on a maximal time interval $[0,T^*)$ with $T^*<\infty$ iff $\lim_{t\to T^*-}\|\mu_t\|_{TV}=\infty$.

The continuity with respect to the initial condition follows from Lemma 6.3 and Gronwall's inequality. Indeed if μ and $\tilde\mu$ are two solutions defined on $[0,T]$ with initial conditions $\mu_0,\tilde\mu_0$ satisfying

$$\|\mu_t\|_{TV},\|\tilde\mu_t\|_{TV}\le R \qquad \text{for } t\in[0,T],$$

then Lemma 6.3 gives

$$\|\mu_t-\tilde\mu_t\|_{BL}\le A(t)+B(t)\int_0^t\|\mu_s-\tilde\mu_s\|_{BL}\,ds$$

with

$$A(t)=e^{C_R^v t}\|\mu_0-\tilde\mu_0\|_{BL}, \qquad B(t)=\{2e^{C_R^v t}(L_R^v\|\mu_0\|_{TV}+\frac{C_R^N L_R^v}{C_R^v}+L_R^N)+L_R^N\}.$$

Since A is non-decreasing, Gronwall's inequality gives

$$\begin{aligned}\|\mu_t - \tilde{\mu}_t\|_{BL} &\leq A(t) + B(t) \int_0^t A(s) e^{\int_s^t B} \, ds \leq A(t) \Big(1 + B(t) \int_0^t e^{\int_s^t B} \, ds \Big) \\ &=: r(t) \|\mu_0 - \tilde{\mu}_0\|_{BL}\end{aligned}$$

which is (12).

7. Corollaries to the Main Result and Remarks

Theorem 3.1 covers a general birth source term of the form

$$N(t, \mu) = \int_{\mathbb{R}^d} \eta(t, x, \mu) \, d\mu(x) \tag{36}$$

where $\eta(t, x, \mu)$ is a measure modeling the birth rate of an individual located at $x \in \mathbb{R}^d$ at time t. Then $N(t, \mu)$ represents the distribution of all the offspring at time t of a population distributed according to a measure μ. This kind of source term has been considered in [10].

Under suitable assumptions on η we can verify that $N(t, \mu)$ is well-defined as a Bochner integral, is continuous in (t, μ) and satisfies (N1) and (N2):

Corollary 7.1. *Suppose that* $\eta : [0, +\infty) \times \mathbb{R}^d \times M_b(\mathbb{R}^d) \to M_b(\mathbb{R}^d)$ *satisfies that for any* $R > 0$ *there exists* $C_R > 0$ *such that for any* $t \geq 0$, $x, \tilde{x} \in \mathbb{R}^d$ *and* $\mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d)$,

- (η1) $\|\eta(t, x, \mu)\|_{TV} \leq C_R$,
- (η2) $\|\eta(t, x, \mu) - \eta(t, x, \tilde{\mu}\|_{BL} \leq C_R \|\mu - \tilde{\mu}\|_{BL}$,
- (η3) $\|\eta(t, x, \mu) - \eta(t, \tilde{x}, \mu\|_{BL} \leq C_R |x - \tilde{x}|$,
- (η4) *the map* $t \to \eta(t, x, \mu)$ *is continuous.*

Then the integral in (36) *is a Bochner integral in* $\overline{M_b}(\mathbb{R}^d)$, *the completion of* $M_b(\mathbb{R}^d)$ *under the BL norm. Moreover, for any* $t \geq 0$ *and any* $\mu \in M_b(\mathbb{R}^d)$,

$$(N(t, \mu), \phi) = \int_{\mathbb{R}^d} (\eta(t, x, \mu), \phi) \, d\mu(x) \qquad \textit{for any } \phi \in L^\infty(\mathbb{R}^d), \tag{37}$$

and $N(t, \mu)$ *is continuous in* (t, μ) *and satisfies (N1) and (N2).*

Proof. We first verify that $N(t, \mu)$ is well-defined as Bochner integral in $\overline{M_b}(\mathbb{R}^d)$ following [22]. We only sketch the proof since more details are given in corollary 4.1. Since $\overline{M_b}(\mathbb{R}^d)$ is separable we have to prove that (i) $\|\eta(t, x, \mu)\|_{BL}$ is $|\mu|$-integrable which is obvious by (η1), and that (ii) the map $x \to \eta(x, t, \mu)$ is weakly measurable i.e., according to [22][Appendix C1], that for any $\phi \in L^\infty(\mathbb{R}^d)$ the map $F(x) := (\eta(t, x, \mu), \phi)$ is measurable. This can be done exactly as in the proof of Corollary 4.1 by considering $F_\varepsilon(x) := (\eta(t, x, \mu), \phi_\varepsilon)$ with $\phi_\varepsilon := \phi * \rho_\varepsilon$ where ρ_ε are the standard mollifiers.

Then F_ε is continuous by $(\eta 3)$ and $F_\varepsilon(x) \to F(x)$ for any $x \in \mathbb{R}^d$ by the dominated convergence theorem. The measurability of F follows.

Relation (37) is proved as in Corollary 4.1. In particular, it follows that

$$\|N(t,\mu)\|_{TV} \le \int_{\mathbb{R}^d} \|\eta(t,x,\mu)\|_{TV}\, d|\mu|(x). \tag{38}$$

Hence, we deduce that N satisfies (N2) using $(\eta 1)$.

We now verify that N satisfies (N1). We have

$$\begin{aligned}
&\|N(t,x,\mu) - N(t,x,\tilde{\mu})\|_{BL} \\
&\quad \le \int_{\mathbb{R}^d} \|\eta(t,x,\mu) - \eta(t,x,\tilde{\mu})\|_{BL}\, d|\mu|(x) + \Big\| \int_{\mathbb{R}^d} \eta(t,x,\tilde{\mu})\, d(\mu-\tilde{\mu})(x) \Big\|_{BL} \\
&\quad =: A + B.
\end{aligned}$$

By $(\eta 2)$ we have

$$A \le \|\mu\|_{TV} C_R \|\mu - \tilde{\mu}\|_{BL} \le R C_R \|\mu - \tilde{\mu}\|_{BL}.$$

To estimate B, take $\phi \in W^{1,\infty}(\mathbb{R}^d)$, $\|\phi\|_{W^{1,\infty}} \le 1$, and denote $F(x) := (\eta(t,x,\tilde{\mu}), \phi)$. Then

$$|F(x)| \le \|\phi\|_\infty \|\eta(t,x,\tilde{\mu})\|_{TV} \le C_R$$

by $(\eta 1)$. Moreover for any $x, \tilde{x} \in \mathbb{R}^d$,

$$|F(x) - F(\tilde{x})| = |(\eta(t,x,\tilde{\mu}) - \eta(t,\tilde{x},\tilde{\mu}), \phi)| \le \|\phi\|_{W^{1,\infty}} \|\eta(t,x,\tilde{\mu}) - \eta(t,\tilde{x},\tilde{\mu})\|_{BL} \le C_R |x - \tilde{x}|$$

by $(\eta 3)$. Thus F is bounded Lipschitz with $\|F\|_{W^{1,\infty}} \le C_R$. Hence

$$\begin{aligned}
|(\int_{\mathbb{R}^d} \eta(t,x,\tilde{\mu})\, d(\mu-\tilde{\mu})(x), \phi)| &= |\int_{\mathbb{R}^d} F(x)\, d(\mu-\tilde{\mu})(x)| \le \|F\|_{W^{1,\infty}} \|\mu - \tilde{\mu}\|_{BL} \\
&\le C_R \|\mu - \tilde{\mu}\|_{BL}.
\end{aligned}$$

Therefore, $B \le C_R \|\mu - \tilde{\mu}\|_{BL}$ and we obtain that N satisfies (N1).

We finally show that N is continuous in (t,μ). To this end, we write

$$\begin{aligned}
\|N(t,\mu) - N(\tilde{t},\tilde{\mu})\|_{BL} &\le \|N(t,\mu) - N(\tilde{t},\mu)\|_{BL} + \|N(\tilde{t},\mu) - N(\tilde{t},\tilde{\mu})\|_{BL} \\
&\le \int_{\mathbb{R}^d} \|\eta(t,x,\mu) - \eta(\tilde{t},x,\mu)\|_{BL}\, d|\mu|(x) + C_R \|\mu - \tilde{\mu}\|_{BL}
\end{aligned}$$

where we used (N1). Moreover, the integral in the right-hand-side goes to 0 as $\tilde{t} \to t$ by the dominated convergence theorem in view of $(\eta 4)$ and $(\eta 1)$ (to bound the integrand by $2C_R$). □

Two examples are worth mentioning. The first one includes mutation by taking $\eta(t,x,\mu) = \eta(x) = \chi(x-y)dy$ where $\chi : \mathbb{R}^d \to \mathbb{R}$. Then for any test-function ϕ,

$$(N(t,\mu), \phi) = \int_{\mathbb{R}^d} (\eta(x), \phi)\, d\mu(x) = \int_{\mathbb{R}^d} \phi(y) \int_{\mathbb{R}^d} \chi(x-y)\, d\mu(x)\, dy = (\chi * \mu, \phi)$$

Hence, $N(t,\mu) = \chi * \mu$. This models the fact that the offsping of an individual located at x is distributed around x according to $\chi(x-y)$. Assuming that χ is integrable and globally Lipschitz function, it is easily seen that assumptions $(\eta 1)-(\eta 4)$ are satisfied.

Another interesting case corresponds to an absence of mutation: the entire offspring of an animal at x stays at x. This can be modeled taking $\eta(t,x,\mu) = \bar{N}(t,\mu)\delta_x$ where $\bar{N}(t,\mu) \in \mathbb{R}$. In this case $N(t,\mu) = \bar{N}(t,\mu)\int_{\mathbb{R}^d} \delta_x \, d\mu(x) = \bar{N}(t,\mu)\mu$. Indeed, we may even consider a slightly more general source term of the form $N(t,\mu) = \bar{N}(t,x,\mu)\mu$.

Consider the special case where the source terms $N^k(t,\mu_t)$, $k = 1,2,3$, have the form of rates namely

$$N^k(t,\mu_t) = \bar{N}^k(t,\cdot,\mu_t)\mu_t^k, \tag{39}$$

where

$$\bar{N}^k : \mathbb{R}_+ \times \mathbb{R}^d \times M_b(\mathbb{R}^d)^3 \to \mathbb{R} \qquad k = 1,2,3.$$

Definition (39) means that for any time $t > 0$ and any $\mu \in M_b(\mathbb{R}^d)^3$, the measure $N^k(t,\mu)$ is defined by

$$(N^k(t,\mu),\phi) = \int_{\mathbb{R}^d} \phi(x)\bar{N}^k(t,x,\mu)\, d\mu(x)$$

for any test-function ϕ. The following corollary shows that under some assumptions on $\bar{N}^k$, the system (8) has a unique solution which is nonnegative if the initial measures μ_0^k are non-negative:

Corollary 7.2. *(Nonnegativity of solutions) Assume that the vector-fields v^i, $i = 1,2,3$, are as in Theorem 3.1 and that the source terms $N^k(t,\mu_t)$ can be written as rates as in* (39)*. Suppose that $\bar{N}^k$, $k = 1,2,3$, are continuous in (t,x,μ) and for any $R > 0$, there exist $L_R^N, C_R^N > 0$ such that for any $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ and any $\mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d)^3$ the following is satisfied:*

(N1') $|\bar{N}^k(t,x,\mu) - \bar{N}^k(t,x,\tilde{\mu})| \leq L_R^N \|\mu - \tilde{\mu}\|_{BL}$,
(N2') $\|\bar{N}^k(t,\cdot,\mu)\|_{W^{1,\infty}} \leq C_R^N$.

Then, given initial condition μ_0^k, $k = 1,2,3$, where each μ_0^k is a non-negative measure, the system (8) *has a unique solution $(\mu_t^1,\mu_t^2,\mu_t^3)$ defined in $[0,T^*)$ where μ_t^k is a non-negative measure for each $k = 1,2,3$ and any $t \in [0,T^*)$.*

Proof. It is easily seen that under the assumptions made on $\bar{N}^k$ that each N^k is continuous and satisfies assumptions (N1) and (N2). We then know by Theorem 3.1 that the system (8) has a unique solution μ_t defined on some time interval $[0,T^*)$.

Letting $c^k(t,x) = \bar{N}^k(t,x,\mu_t)$ the system becomes

$$\begin{aligned} &\partial_t \mu_t^1 + \nabla \cdot (v^1[\mu_t]\mu_t^1) = c^1(t,x)\mu_t^1, \\ &\partial_t \mu_t^2 - \Delta \mu_t^2 = c^2(t,x)\mu_t^2, \\ &\partial_t \mu_t^3 + \nabla \cdot (v^2[\mu_t]\mu_t^3) = c^3(t,x)\mu_t^3. \end{aligned} \tag{40}$$

Direct calculations show that the solutions μ_t^1, μ_t^2 and μ_t^3 are given by

$$\mu_t^k = \exp\Big(\int_0^t c^k(s, \mathcal{T}_{t,s}^{v^k[\mu]}(.))\,ds\Big)(\mathcal{T}_{0,t}^{v^k[\mu]} \sharp \mu_0^k), \quad k = 1, 3, \tag{41}$$

where $\mathcal{T}_{s,t}^{v^k}$ is the flow associated with the vector-field $v^k[\mu]$, and

$$\mu_t^2 = \exp\Big(\int_0^t c^2(s)\,ds\Big)(K_t * \mu_0^2).$$

Here (41) means that for any test-function ϕ,

$$\int_{\mathbb{R}^d} \phi\, d\mu_t^k = \int_{\mathbb{R}^d} \phi(\mathcal{T}_{0,t}^{v^k[\mu]}(x))\exp\Big(\int_0^t c^k(s, \mathcal{T}_{0,s}^{v^k[\mu]}(x))\,ds\Big)\,d\mu_0^k$$

(where we used the fact that $\mathcal{T}_{t,s} \circ \mathcal{T}_{0,t} = \mathcal{T}_{0,s}$). Hence, it follows that if μ_0^k, $k = 1, 2, 3$, are non-negative measures then μ_t^k, $k = 1, 2, 3$, are non-negative for any $t \geq 0$. $\square$

Remark 7.1. *From Corollary 7.2 and its proof, it is clear that if μ_0^k, $k = 1, 2, 3$, are probability measures and there are no source terms in system* (8) *(i.e., $N^k = 0$, $k = 1, 2, 3$), then for any $t \in [0, T]$, μ_t^k, $k = 1, 2, 3$, are also probability measures.*

We now provide some conditions ensuring that the solution given by Theorem 3.1 defined in $[0, T^*)$ is indeed global, i.e., $T^* = +\infty$.

Corollary 7.3. *(Global existence of solutions) Assume that assumptions (V1) and (N1) hold and that (V2) and (N2) are replaced by assumptions (V2") and (N2") given by*

- (V2") $Lip(v^i[\mu]) \leq C$, $i = 1, 2$,
- (N2") $\|N^k(t,\mu)\|_{TV} \leq C(1 + \|\mu\|_{TV})$ *for* $k = 1, 2, 3$ *and* $t \in \mathbb{R}$,

where the constant C is independent of μ. Then, for any initial condition $\mu_0 \in M_b(\mathbb{R}^d)^3$, system (8) *has a unique solution which is defined on* $[0, +\infty)$.

Proof. Notice first that (N2") implies (N2) and also that (V2") implies (V2) (the fact that $\|v^i[\mu]\|_\infty \leq C_R^v$ follows from (V1) with $\tilde{\mu} = 0$). Given initial conditions μ_0, we then know that there exists a solution defined over a maximal interval time $[0, T^*)$.

Suppose $T^* < +\infty$. Then we know that $\|\mu_t\|_{TV} \to +\infty$ as $t \to T^{*-}$. Notice that the vector-field $(t,x) \to v^i[\mu_t](x)$ is continuous and globally Lipschitz by (V2") so that the measures $\mathcal{T}_t^{v^1[\mu]}\sharp\mu_0^1$ and $\mathcal{T}_t^{v^2[\mu]}\sharp\mu_0^3$ exist for all time. Since $\mu_t = \Gamma(\mu)_t$ (the definition of Γ is given in Section 6) we have

$$\begin{aligned}
\|\mu_t\|_{TV} &= \sum_k \|\Gamma^k(\mu)_t\|_{TV} \\
&\leq \|\mathcal{T}_t^{v^1[\mu]}\sharp\mu_0^1\|_{TV} + \|\mathcal{T}_t^{v^2[\mu]}\sharp\mu_0^3\|_{TV} + \|P_t\mu_0^2\|_{L^1} \\
&\quad + \int_0^t \|\mathcal{T}_{s,t}^{v^1[\mu]}\sharp N^1(s,\mu_s)\|_{TV} + \|\mathcal{T}_{s,t}^{v^2[\mu]}\sharp N^3(s,\mu_s)\|_{TV} + \|P_{t-s}N^2(s,\mu_s)\|_{L^1}\,ds \\
&\leq \|\mu_0\|_{TV} + \int_0^t \sum_k \|N^k(s,\mu_s)\|_{TV}\,ds \\
&\leq \|\mu_0\|_{TV} + 3Ct + 3C\int_0^t \|\mu_s\|_{TV}\,ds.
\end{aligned}$$

Gronwall inequality gives

$$\|\mu_t\|_{TV} \leq (\|\mu_0\|_{TV} + 3Ct)e^{3Ct} \leq (\|\mu_0\|_{TV} + 3CT)e^{3CT}$$

so that $\|\mu_t\|_{TV}$ is bounded near T^*, a contradiction. $\square$

Remark 7.2. *Source terms satisfying conditions in Corollaries 7.2 and 7.3 arise in biologically relevant applications including the following classical source term in population dynamics:*

a. *Holling type functions* [32] *where, for example, for* $k = 1,2,3$

$$\bar{N}^k(t,x,\mu) = \bar{N}^k(\mu) = \frac{1}{1 + \gamma^k \sum_{j=1}^3 (\int_{\mathbb{R}^d} w^j(y)d\mu^j(y))_+}$$

for $w \in W^{1,\infty}(\mathbb{R}^d)$ *and a nonnegative constant* γ^k. *Here,* $x_+ = \max\{x, 0\}$.

Clearly, $|\bar{N}^k(\mu)| \leq 1$. *Thus, (N2') and (N2") hold. To show that (N1') holds, using the notation* $(\mu^j, w^j) = \int_{\mathbb{R}^d} w^j(y)d\mu^j(y))$ *and that* $|x_+ - y_+| \leq |x-y|$ *for any* $x, y \in \mathbb{R}$, *we have for any* $\mu, \tilde{\mu} \in M_b(\mathbb{R}^d)$ *that*

$$\begin{aligned}
|\bar{N}^k(\mu) - \bar{N}^k(\tilde{\mu})| &\leq \frac{\gamma^k \sum_{j=1}^3 |(\mu^j, w^j)_+ - (\tilde{\mu}^j, w^j)_+|}{(1 + \gamma^k \sum_{j=1}^3 (\mu^j, w^j)_+)(1 + \gamma^k \sum_{j=1}^3 (\tilde{\mu}^j, w^j)_+)} \\
&\leq \gamma^k \sum_{j=1}^3 |(\mu^j, w^j) - (\tilde{\mu}^j, w^j)| \\
&\leq \gamma^k \|w\|_{W^{1,\infty}} \|\mu - \tilde{\mu}\|_{BL}.
\end{aligned}$$

This establishes the continuity of $\bar{N}^k$ *and assumption (N1').*

b. *Ricker type functions* [46] *where, for example, for* $k = 1,2,3$

$$\bar{N}^k(t,x,\mu) = \bar{N}^k(\mu) = \exp\Big(-\gamma^k \sum_{j=1}^3 \Big(\int_{\mathbb{R}^d} w^j(y)d\mu^j(y)\Big)_+\Big),$$

for $w \in W^{1,\infty}(\mathbb{R}^d)$ *and a nonnegative constant* γ^k.

Clearly, $|\bar{N}^k(\mu)| \leq 1$. *Hence, (N2') and (N2") holds. To show that (N1') holds, let* $\mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d)$. *For any* $x \in \mathbb{R}$ *between* $-\gamma^k \sum_{j=1}^3 (\mu^j, w^j)_+$ *and* $-\gamma^k \sum_{j=1}^3 (\tilde{\mu}^j, w^j)_+$, *i.e.,*

$$x = -\theta\gamma^k \sum_{j=1}^3 (\mu^j, w^j)_+ - (1-\theta)\gamma^k \sum_{j=1}^3 (\tilde{\mu}^j, w^j)_+, \qquad \theta \in [0,1],$$

we have

$$\begin{aligned} |x| &\leq \gamma^k \sum_{j=1}^3 |(\mu^j, w^j)| + |(\tilde{\mu}^j, w^j)| \leq \gamma^k \sum_{j=1}^3 \|w^j\|_\infty (\|\mu^j\|_{TV} + \|\tilde{\mu}^j\|_{TV}) \\ &\leq 2R\gamma^k \sum_{j=1}^3 \|w^j\|_\infty \end{aligned}$$

Thus

$$\begin{aligned} |\bar{N}^k(\mu) - \bar{N}^k(\tilde{\mu})| &\leq \exp\Big(2R\gamma^k \sum_{j=1}^3 \|w^j\|_\infty\Big)\gamma^k \sum_{j=1}^3 |(\mu^j, w^j)_+ - (\tilde{\mu}^j, w^j)_+| \\ &\leq L_N^R \|\mu - \tilde{\mu}\|_{BL}. \end{aligned}$$

Hence, assuming v^i, $i = 1, 2$, *satisfy (V2"), then our results guarantee the existence of a global nonnegative solution for such source terms. Also, since the solution,* μ^j, $j = 1, 2, 3$, *is nonnegative then* $(\mu^j, w^j)_+ = (\mu^j, w^j)$.

We now give an example of a family of vector-fields satisfying assumptions (V1) and (V2) relevant in applications.

Remark 7.3. *Keeping in mind the discussion leasding to the system (5) with the vector field (6), it is natural to consider vector fields* v^1 *and* v^2 *of the form*

$$v^k[\mu](x) = \int_{\mathbb{R}^d} K(x,y)\, d\mu_1(y) + \int_{\mathbb{R}^d} \tilde{K}(x,y)\, d\mu_3(y) + T[\mu_2](x)$$

where $K, \tilde{K} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $T[\mu^2] : \mathbb{R}^d \to \mathbb{R}^d$. *If* $K, \tilde{K}$ *are bounded and globally Lipschitz then the first two integrals satisfy assumptions (V1) and (V2). We now want to define* $T[\mu_2](x)$ *so as to model chemotaxis. In a smooth setting we may let* $T[\mu_2](x) \simeq \nabla\mu_2$ *to account for the fact that the movement of cells is driven by the gradient of the chemical. Since this is not adequate in our measure-valued setting, we approximate* $\nabla\mu^2$ *using an idea presented in* [34]. *Indeed the authors in* [34] *introduced a non-local gradient of a function* f *by*

$$\overset{\circ}{f}(x) = \frac{d}{|S^{d-1}(0,1)|\varepsilon} \int_{S^{d-1}(0,1)} f(x+\varepsilon y) y\, d\sigma(y) \tag{42}$$

for $\varepsilon > 0$. This is indeed an approximation of ∇f when f is C^1 since in the limit as $\varepsilon \to 0$ we have $\overset{\circ}{f}(x) = \nabla f(x) + O(\varepsilon)$. Let us rewrite (42) *as*

$$\overset{\circ}{f}(x) = \frac{d}{|S^{d-1}(0,1)|\varepsilon^{d+1}} \int_{S^{d-1}(x,\varepsilon)} (z-x) f(z)\, d\sigma_{S^{d-1}(x,\varepsilon)}(z).$$

Let $g_\delta \in C^\infty(\mathbb{R}^d)$ be radial nonnegative such that $g_\delta(z)dz \to \sigma_{S^{d-1}(0,1)}$ as $\delta \to 0$. Then $g_\delta(\frac{z-x}{\varepsilon})\frac{1}{\varepsilon}dz \to \sigma_{S^{d-1}(x,\varepsilon)}$ as $\delta \to 0$. We thus have

$$\overset{\circ}{f}(x) \simeq \frac{d}{|S^{d-1}|\varepsilon^{d+2}} \int_{\mathbb{R}^d} (z-x) g_\delta(\frac{z-x}{\varepsilon}) f(z)\, dz = \int_{\mathbb{R}^d} K_{\delta,\varepsilon}(x,z)\, d\mu_2(z)$$

where $\mu_2(z) = f(z)dz$ and $K_{\delta,\varepsilon}(x,z) = \frac{d}{|S^{d-1}|\varepsilon^{d+2}}(z-x)g_\delta(\frac{z-x}{\varepsilon})$. We thus propose the following as an approximate non-local gradient of the measure μ_2:

$$\overset{\circ}{\mu_2}(x) = \int_{\mathbb{R}^d} K_{\delta,\varepsilon}(x,z)\, d\mu_2(z).$$

We can then take $T[\mu_2](x) = C\times \overset{\circ}{\mu_2}(x)$, where C is a constant. Assume for example that g_δ has compact support for any positive δ. Then given positive ε and δ, $K_{\delta,\varepsilon}$ is bounded and globally Lipschitz, and so T satisfies (V1)-(V2).

We end this section with a brief comment concerning the numerical simulation of a system like (8). This is an important issue from the point of view of applications. Few numerical schemes have been studied in the framework of measure-valued solutions, and all of them deal with a single transport equation with the exception of the recent paper [11]. We mention the Escalator Box Train method [5],[11],[26], Particle-Splitting method [12],[29], and a recently proposed finite difference method [2] which shows interesting properties in terms of velocity of convegence and accuracy.

Finally, we point out that under the assumption of Theorem 3.1, the solution μ^2 of the diffusion equation is more regular than a mere measure due to the regularizing effect of the heat operator:

Remark 7.4. *It follows from the explicit expression* (17) *that the solution u of the diffusion equation* (15) *satisfies $u \in L^\infty(0,T;L^1(\mathbb{R}^d))$ for any $T > 0$. Moreover for any $p \geq 1$ and any $t > 0$,*

$$\|K_t\|_p \leq Ct^{-\frac{d}{2}(1-1/p)} \qquad \text{and} \qquad \|\partial_{x_i} K_t\|_p \leq Ct^{-\frac{d}{2}(1-1/p)-1/2}.$$

It then follows that $u \in L^s(0,T;W^{1,p}(\mathbb{R}^d))$ for any $s,p \geq 1$ such that $\frac{2}{s} + \frac{d}{p} > d+1$ with

$$\|u\|_{L^s(0,T;W^{1,p}(\mathbb{R}^d))} := \Big(\int_0^T \|u(t,\cdot)\|^s_{W^{1,p}(\mathbb{R}^d)}\, dt\Big)^{1/s} \leq C(\|u_0\|_{TV} + \|\sigma\|_{TV}). \tag{43}$$

To apply this estimate to μ^2, notice first that since for any $T < T^*$ there exists $R > 0$ such that $\max_{t\in 0,T]} \|\mu_t\|_{TV} \leq R$, we have $\max_{t\in 0,T]} \|N^2(t,\mu_t)\|_{TV} \leq R'$ thanks *to assumption (N2). Thus the measure* $\sigma := \int_0^T \delta_s \otimes N^2(s,\mu_s)\,ds$ *(see equation* (19)*) is bounded. It then follows from* (43) *that* $\mu^2 \in L^s(0,T;W^{1,p}(\mathbb{R}^d))$ *for any* $s,p \geq 1$ *such that* $\frac{2}{s}+\frac{d}{p} > d+1$ *and for any* $T < T^*$.

Acknowledgement: The authors would like to thank two anonymous referees for useful suggestions that led to an improved version of the manuscript.

Appendix

Here we give a brief sketch of the proof of Proposition 4.1.

Sketch of Proof. We first verify that $u(t,x) = (P_t u_0)(x) + (K * \sigma)(t,x)$ is a solution of (15). Notice that $u \in L^1(Q_T)$ for any $T > 0$ since

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |u(t,x)|\,dtdx &\leq \int_0^T \int_{\mathbb{R}^d} P_t|u_0|(x)\,dxdt + \int_0^T \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(t-s,x-y)dxd|\sigma|(s,y)dt \\ &\leq T(|u_0|(\mathbb{R}^d) + |\sigma|((0,T)\times\mathbb{R}^d)),\end{aligned}$$

were we used that for any $t>0$, $\|K(t,\cdot)\|_1 = 1$ and $\|P_t u_0\|_1 \leq |u_0|(\mathbb{R}^d)$.

We can then verify that u verifies (18), and thus solves (15). Indeed this follows from the fact that for any $T>0$,

$$\int_{\mathbb{R}^d} \phi(T,x)\tilde{u}(T,x)dx = \int_0^T \int_{\mathbb{R}^d} (\partial_s + \Delta)\phi(s,x)\tilde{u}(s,x)dsdx + \int_0^T \int_{\mathbb{R}^d} \phi(s,x)\,d\sigma(s,x), \tag{44}$$

where $\tilde{u}(t,x) := (K*\sigma)(t,x) = \int_0^t \int_{\mathbb{R}^d} K(t-s,x-y)\,d\sigma(s,y)$, and

$$\int_{\mathbb{R}^d} \phi(T,x)P_t u_0(x)dx - \int \phi(0,x)\,du_0(x) = \int_0^T \int_{\mathbb{R}^d} (\partial_s+\Delta)\phi(s,x)P_s u_0(x)dsdx. \tag{45}$$

Both (44) and (45) are quite standard to prove.

We now verify that $u(t)$ satisfies the intial condition in the measure sense:

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} \phi(x)u(t,x)dx = \int_{\mathbb{R}^d} \phi\,du_0 \qquad \text{for any } \phi \in C_b(\mathbb{R}^d) \tag{46}$$

if $|\sigma|(\{0\}\times\mathbb{R}^d) = 0$. This follows from

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \phi(x)P_t u_0(x)dx = \int \phi\,du_0 \qquad \text{and} \qquad \lim_{t\to 0}\int_{\mathbb{R}^d} \phi(x)\tilde{u}(t,x)dx = 0. \tag{47}$$

Indeed on the one hand,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} \phi(x)\tilde{u}(t,x)dx\Big| &\leq \|\phi\|_\infty \int_{\mathbb{R}^d}\int_0^t \Big(\int_{\mathbb{R}^d} K(t-s,x-y)dx\Big)d|\sigma|(s,y) \\ &\leq \|\phi\|_\infty |\sigma|((0,t)\times\mathbb{R}^d)\end{aligned}$$

which goes to $\|\phi\|_\infty|\sigma|(\{0\}\times\mathbb{R}^d)=0$ as $t\to 0$. On the other hand the first limit in (47) follows writing $\int_{\mathbb{R}^d}\phi(x)P_tu_0(x)dx=\int_{\mathbb{R}^d}P_t\phi\,du_0$ and passing to the limit using the dominated convergence theorem noticing that (i) since $\phi\in C_b(\mathbb{R}^d)$, we have $\lim_{t\to 0}P_t\phi(x)=\phi(x)$ for any $x\in\mathbb{R}^d$ (see e.g. [21]), and (ii) $\|P_t\phi\|_\infty\le\|\phi\|_\infty$.

We finally show that the measure $u(t,x)dx$ is continuous in t for the weak convergence, i.e.,

$$\lim_{t'\to t}\int_{\mathbb{R}^d}\phi(x)u(t',x)dx=\int_{\mathbb{R}^d}\phi(x)u(t,x)dx \tag{48}$$

for any $\phi\in C_b(\mathbb{R}^d)$ and any $t>0$ such that $|\sigma|(\{t\}\times\mathbb{R}^d)=0$. First

$$\int_{\mathbb{R}^d}\phi P_{t'}u_0\,dx=\int_{\mathbb{R}^d}P_{t'}\phi\,du_0\to\int_{\mathbb{R}^d}P_t\phi\,du_0=\int_{\mathbb{R}^d}\phi P_tu_0\,dx$$

where we pass to the limit using the Dominated Convergence Theorem as before. Moreover, assuming w.l.o.g. $t<t'$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\phi(x)\tilde{u}(t',x)dx-\int_{\mathbb{R}^d}\phi(x)\tilde{u}(t,x)dx\\&=\int_0^t\int_{\mathbb{R}^d}P_{t'-s}\phi(y)-P_{t-s}\phi(y)\,d\sigma(s,y)+\int_t^{t'}\int_{\mathbb{R}^d}P_{t'-s}\phi(y)\,d\sigma(s,y).\end{aligned} \tag{49}$$

The second term on the right-hand side can be bounded by

$$\|P_{t'-s}\phi\|_\infty|\sigma|([t,t']\times\mathbb{R}^d)\le\|\phi\|_\infty|\sigma|([t,t']\times\mathbb{R}^d). \tag{50}$$

Independently, for any $0<\delta<R<\infty$, it is easily seen that there exists $L_{\delta,R}>0$ such that for any $\phi\in L^\infty$ and $t',t\in[\delta,R]$,

$$\|P_{t'}\phi-P_t\phi\|_\infty\le L_{\delta,R}\|\phi\|_\infty|t'-t|.$$

We then bound the first term in (49) as follows

$$\begin{aligned}&\int_0^{t-\delta}\int_{\mathbb{R}^d}|P_{t'-s}\phi(y)-P_{t-s}\phi(y)|\,d|\sigma|(s,y)+\int_{t-\delta}^t\int_{\mathbb{R}^d}|P_{t'-s}\phi(y)-P_{t-s}\phi(y)|\,d|\sigma|(s,y)\\&\le L_{\delta,R}|t-t'||\sigma|(Q)+2\|\phi\|_\infty|\sigma|([t-\delta,t]\times\mathbb{R}^d)\\&\le L_{\delta,R}|t-t'||\sigma|(Q)+o_\delta(1),\end{aligned}$$

where $o_\delta(1)\to 0$ as $\delta\to 0$ uniformly in t'. Plugging this last inequality and (50) into (49), we deduce (48) taking first take δ small enough and then $t'\to t$. Thus $u(t,x)=(P_tu_0)(x)+(K*\sigma)(t,x)$ is a solution of (15).

We now prove the uniqueness using an idea from [7]. To this end, it suffices to verify that the zero function is the only solution to (15) with $u_0=\sigma=0$. Let u be such a solution, in particular $u\in L^1(Q_T)$ for any $T>0$. In view of (16),

$$\iint_{Q_T}u(t,x)(\partial_t+\Delta)\phi(t,x)\,dtdx=0\qquad\text{for any }\phi\in C_c^{1,2}([0,T]\times\mathbb{R}^d),\ \phi(T,\cdot)=0.$$

We can in fact take any $\phi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ with bounded derivatives and $\phi(T,\cdot)=0$. Given $\zeta \in C_c^\infty(Q_T)$ let ϕ be the solution to

$$(\partial_t+\Delta)\phi = \zeta \qquad \text{in } Q_T \qquad \text{with } \zeta(T,\cdot)=0$$

given by $\phi(t,x)=(K*\tilde{\zeta})(t,x)$, $\tilde{\zeta}(t,x)=-\zeta(T-t,x)$. Then $\phi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ with bounded derivatives and $\phi(T,\cdot)=0$. It follows that $\iint_{Q_T} u\zeta\, dtdx = 0$ for any $\zeta \in C_c^\infty(Q_T)$, so that $u=0$ in Q_T.

References

[1] A.S. Ackleh, J. Cleveland, H.R. Thieme, *Population dynamics under selection and mutation: long-time behavior for differential equations in measure spaces*, Journal of Differential Equations, 261 (2016), 1472-1505.

[2] A.S. Ackleh, R. Lyons, and N. Saintier, *Finite Difference Schemes for a Structured Population Model in the Space of Measures*, Mathematical Biosciences and Engineering, 17 (2020), 747-775.

[3] G. Aletti, G. Naldi, G. Toscani, *First-order continuous models of opinion formation*, SIAM Journal of Applied Mathematics, 67 (2007), 837-853.

[4] N. Bellomo, *Modeling Complex Living Systems A Kinetic Theory and Stochastic Game Approach*. Birkhäuser, (2008).

[5] A. Brannstrom, L. Carlsson, and D. Simpson, *On the convergence of the escalator boxcar train*, SIAM J. Num. Anal., 51 (2013), 3213-3231.

[6] W. Braun and K. Hepp, *The Vlasov Dynamics and Its Fluctuations in the 1/N Limit of Interacting Classical Particles*, Communications in Mathematical Physics, 56 (1977), 101-113.

[7] H. Brezis, A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, MRC technical summary report 2277 (1981).

[8] J.A. Cañizo, J.A. Carrillo, S. Cuadrado, *Measure solutions for some models in population dynamics*, Acta Applicandae Mathematicae, 123 (2013), 141-156 .

[9] J.A. Cañizo, J.A. Carrillo, J. Rosado, *A well-posedness theory in measures for some kinetic models of collective motion*, Mathematical Models and Methods in Applied.Sciences, 21 (2011), 515-539.

[10] J.A. Carrillo, R.M. Colombo, P. Gwiazda, A. Ulikowska, *Structured populations, cell growth and measure valued balance laws*, Journal of Differential Equations, 252 (2012), 3245-3277.

[11] J.A. Carrillo, P. Gwiazda, K. Kropielnicka, and A.K. Marciniak-Czochra, Anna K, *The escalator boxcar train method for a system of age-structured equations in the space of measures*, SIAM J. Numer. Anal., 57 (2019), no. 4, 1842-1874.

[12] J.A. Carrillo, P. Gwiazda, A. Ulikowska, *Splitting-particle methods for structured population models: convergence and applications.*, Math. Models Methods Appl. Sci., 24 (2014), 2171-2197.

[13] J. Cleveland and A.S. Ackleh, *Evolutionary Game Theory on Measure Spaces: Well-Posedness*, Nonlinear Analysis: Real World Applications, 14(2013), 785-797.

[14] A. Colombi, M. Scianna, L. Preziosi, *A measure-theoretic model for collective cell migration and aggregation*, Mathematical Modelling of Natural Phenomena, 10 (2015), 32-63.

[15] A Colombi, M. Scianna, A. Tosin, *Differentiated cell behaviour: a multiscale approach using measure theory*, Journal of mathematical biology, 71 (2015), 1049-1079.

[16] G. Crippa, and M. Lécureux-Mercier, *Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow*, Nonlinear Differ. Equ. Appl., 20 (2013), 523-537.

[17] J. Diestel, J.J. Uhl, Jr.,*Vector measures*, Mathematical Surveys and Monographs, Vol.15, American Mathematical Society (1977).

[18] B. During, P. Markowich, J. F. Pietschmann, M-T. Wolfram, *Boltzmann and Fokker-Planck equations modelling opinion formation in the presence of strong leaders*, Proceedings of the Royal Society A, 465 (2009).

[19] R. Dobrushin, *Vlasov equations*, Functional Analysis and Applications, 13 (1979), 115-123.
[20] R.M. Dudley, *Convergence of Baire measures*, Studia Mathematika, T. XXVII (1966), 251-268.
[21] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, American Mathematical Society (2010).
[22] J.H.M. Evers, S. C. Hille, A. Muntean, *Mild solutions to a measure-valued mass evolution problem with flux boundary conditions*, Journal of Differential Equations, 259 (2015), 1068-1097.
[23] J.H.M. Evers, S. C. Hille, A. Muntean, *Measure-valued mass eolution problems with flux boundary conditions and solutions-dependent velocities*, SIAM Journal on Mathematical Analysis, 48 (2016), 1929-1953.
[24] M. Di Francesco, S. Fagioli, *Measure solutions for non-local interaction: PDEs with two species*, Nonlinearity, 26 (2013), 2777-2808.
[25] F. Golse, *The mean-field limit for the dynamics of large particle systems*, Journées équations aux dérivées partielles, 9 (2003), 1-47.
[26] P. Gwiazda, J. Jablonski, A. Marciniak-Czochra, and A. Ulikowska, *Analysis of particle methods for structured population models with nonlocal boundary term in the framework of bounded Lipschitz distance*, Numer. Methods Partial Differential Equations, 30 (2014), 1797-1820.
[27] P. Gwiazda, T. Lorenz, A. Marciniak Czochra, *A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients*, Journal of Differential Equations, 248 (2010), 2703-2735.
[28] P. Gwiazda, A. Marciniak Czochra, *Structured population equations in metric spaces*, Journal of Hyperbolic Differential Equations, 7 (2010), 733-773.
[29] P. Gwiazda, P. Orlinski, and A. Ulikowska, *Finite range method of approximation for balance laws in measure spaces.*, Kinet. Relat. Models., 10 (2017), no. 3, 669-688.
[30] S.C. Hille, T. Szarek, D.T.H. Worm, M.A. Ziemlanska, *On a Schur-like property for spaces of measures*, arxiv.org/pdf/1703.00677.pdf
[31] S.C. Hille, D. Worm, *Embedding of semigroups of Lipschitz maps into positive linear semigroups on ordered Banach spaces generated by measures*, Integral Equations and Operator Theory, 63 (2009), 351-371.
[32] C.S. Holling, *The components of predation as revealed by a study of small-mammal predation of the European pine sawfly*, The Canadian Entomologist, 91 (1959), 293-320.
[33] J.M. Holte, Discrete Gronwall lemma and applications,
http://homepages.gac.edu/ holte/publications/GronwallLemma.pdf.
[34] T. Hillen, H. Othmer, *The diffusion limit of transport equations II: chemotaxis equations*, SIAM Journal on Applied Mathematics, 62 (2002), 1222-1250
[35] T. Lorenz, *Mutational Analysis*, Lecture Notes in Mathematics 1996, (2010).
[36] H. Neunzert, *The Vlasov equation as a limit of Hamiltonian classical mechanical systems of interacting particles*, Trans. Fluid Dynamics, 18 (1977), 663-678.
[37] H. Neunzert, *An introduction to the nonlinear Boltzmann-Vlasov equation*, in Kinetic theories and the Boltzmann equation (Montecatini, 1981), pp. 60-110, Lecture Notes in Math. 1048, Springer (1984).
[38] L. Pareschi & G. Toscani, *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*, Oxford University Press, Oxford (2014).
[39] L. Pedraza, J.P. Pinasco, N. Saintier, *Measure-valued opinion dynamics*, M3AS: Mathematical Models and Methods in Applied Sciences, 01(30) 2020.
[40] M. Perez-Llanos, J.P. Pinasco, N. Saintier, *Opinion attractiveness and its effect in opinion formation models*, submitted.
[41] M. Pérez-Llanos, J.P. Pinasco, N. Saintier, and A. Silva, *Opinion formation models with heterogeneous persuasion and zealotry*, SIAM Journal on Mathematical Analysis, 50 (2018), 4812-4837.
[42] B. Piccoli, and F. Rossi *Generalized Wasserstein Distance and its Application to Transport Equations with Source*, Arch. Rational Mech. Anal., 211 (2014), 335-358.
[43] B. Piccoli, and F. Rossi *On Properties of the Generalized Wasserstein Distance*, Arch. Rational Mech. Anal., , 222 (2016), 1339-1365.

[44] J.P. Pinasco, N. Saintier, F. Vazquez, *The role of voting intention in public opinion polarization*, Phys. Rev. E 101, 012101 - Published 2 January 2020.
[45] J.P. Pinasco, N. Saintier, F. Vazquez, *A model for a phase transition between political mono-polarization and bi-polarization*, submitted.
[46] W.E. Ricker, W. E., *Stock and Recruitment*, Journal of the Fisheries Research Board of Canada, 11 (1954), 559-623.
[47] M. Scianna, A. Tosin, L. Preziosi, *From discrete to continuous models of cell colonies: A measure-theoretic approach*, arXiv:1108.1212, 2011.
[48] H. Spohn, *Large scale dynamics of interacting particles*, Texts and Monographs in Physics, Springer 1991.
[49] G. Toscani, *Kinetic models of opinion formation*, Communications in Mathematical Sciences, 4 (2006), 481-496.
[50] A. Ulikowska, *An age-structured, two-sex model in the space of radon measures: Well posedness*, Kinet. Relat. Models, 5 (2012), 873-900.
[51] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics 58, American Mathematical Society, (2003).

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, Louisiana 70504, U.S.A.

Email address: ackleh@louisiana.edu

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires (1428) Pabellón I - Ciudad Universitaria - Buenos Aires - ARGENTINA

Email address: nsaintie@dm.uba.ar